Lyapunov theory in analysis of nonlinear partial differential equations

Seminar talk

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July 18, 2012

Outline of the talk

- Introduction
  - Lypunov stability theory
  - Examples of PDEs
  - PDEs as dynamical systems
- Lyapunov theory for nonlinear evolution equations
  - State of the art
  - Outline of my thesis
- Discussion on a role of PDE theory in ACROSS project

Given dynamical system

$$\dot{x} = f(x)\,, \quad x(0) = x_0,$$

$x \in \mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}^d$ – smooth vector function.

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Theorem (Lyapunov stability)

Let $x_E = 0$ be an equilibrium of the system and $\mathcal{U} \subset \mathbb{R}^d$. Let $V : \mathcal{U} \to \mathbb{R}$ be positive definite and $\dot{V}(x(t))$ negative semidefinite, then x_E is stable. Moreover, if $\dot{V}(x(t))$ is negative definite, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) + cW(x(t)) \leq 0 \quad \textit{for all } t > 0$$

and for some positive definite function $W : \mathcal{U} \to \mathbb{R}$ and constant $c > 0$, then x_E is asymptotically stable.

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Theorem (Exponential convergence)

Assume in addition $V(x) \leq c' W(x)$ and V is quadratic ($V(x) = x^T \mathbf{V} x$), then

$$V(x(t)) \leq V(x_0) e^{-ct/c'}, \ t > 0 \quad \textit{and } x(t) \textit{ converges exponentially to } x_E.$$

- Heat equation

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \kappa \Delta u$$

$u(t; x, y, z)$ – temperature, $\kappa > 0$

Introduction

Examples of partial differential equations

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- Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}.$$

Introduction

PDEs as dynamical systems

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$$\dot{\tilde{u}}(t) = A\tilde{u}(t),$$

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Other evolution equations (nonlinear):

$$\frac{\partial u}{\partial t} = \Delta(u^m), \quad \text{porus medium equation,}$$

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad \text{parabolic } p\text{-Laplace equation,}$$

$$\frac{\partial u}{\partial t} = \operatorname{div}(u^{\beta}\nabla\Delta u), \quad \text{thin-film equation.}$$

For a given evolution equation, a nonnegative functional E is called an entropy (Lyapunov functional) if it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} E[u(t)] + cQ[u(t)] \le 0 \quad \text{along each solution trajectory} \quad t \mapsto u(t), \qquad \text{(EPI)}$$

for some nonnegative functional Q and constant $c \ge 0$.
Typical choice of entropies:

- α-functionals ($\alpha \in \mathbb{R}$):

$$E_\alpha[u] = \frac{1}{\alpha(\alpha-1)} \int_\Omega (u^\alpha - \alpha u + \alpha - 1)\mathrm{d}x, \quad E_1[u] = \int_\Omega (u \log u - u + 1)\mathrm{d}x \,.$$

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Example. *Heat equation $\partial_t u = \Delta u$ on $\mathbb{T}^d \times (0,\infty)$.*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2[u] + \int_{\mathbb{T}^d} |\nabla u|^2 \mathrm{d}x = 0 \;\rightsquigarrow\; E_2[u(t)] + \int_0^t \int_{\mathbb{T}^d} |\nabla u|^2 \mathrm{d}x\,\mathrm{d}s = E_2[u_0], \;\; t > 0.$$

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Applications of entropy production inequalities:

- existence of solutions, long-time behaviour, positivity, numerical schemes, ...

- ongoing miniaturisation trend in nanotechnology

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Quantum Drift-Diffusion models

1. Density gradient model

$$\partial_t u = T\Delta u + \operatorname{div}\left(u\nabla(V_B[u] + V)\right),$$
$$-\lambda^2 \Delta V = u - C_{dot}, \quad V_B[u] = -\frac{\hbar^2}{2}\frac{\Delta\sqrt{u}}{\sqrt{u}}.$$

[u - particle density, T - temperature, V - electrostatic potential, V_B - Bohm potential, C_d - doping profile, λ - Debye length, $\hbar$ - scaled Planck constant; Ref: Ancona et al. '89., Pinnau '00.]

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2. Nonlocal quantum drift-diffusion model

$$\partial_t u = \operatorname{div}(u\nabla(A + V)),$$

$$u(t;x) = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} \operatorname{Exp}\left(A(t;x) - \frac{|p|^2}{2}\right)\mathrm{d}p, \quad x \in \mathbb{R}^d, \ t > 0.$$

[A - quantum chemical potential, $\operatorname{Exp}(a) = W(\exp(W^{-1}(a)))$ - quantum exponential; Ref: Degond et al. '05.]

Approximation of A in terms of the scaled Planck constant $\hbar$ (pseudo-differential calculus)

$$A = \log u - \frac{\hbar^2}{6}\frac{\Delta\sqrt{u}}{\sqrt{u}} + \frac{\hbar^4}{360}\sum_{i,j=1}^{d}\Big(\frac{1}{2}(\partial_{ij}^2 \log u)^2 + \frac{1}{u}\partial_{ij}^2(u\partial_{ij}^2 \log u)\Big) + O(\hbar^6).$$

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Model equations:

1. $O(\hbar^4)$: fourth-order quantum diffusion equation (also known as Derrida-Lebowitz-Speer-Spohn (DLSS) equation)

$$\partial_t u + \operatorname{div}\left(u\nabla\left(\frac{\Delta\sqrt{u}}{\sqrt{u}}\right)\right) = 0\,. \qquad \text{(DLSS)}$$

[Toom model: $\partial_t u + \frac{1}{2}(u(\log u)_{xx})_{xx} = 0$, Bleher et al. '94., Jüngel and Matthes '08., Savaré et al. '09.]

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2. $O(\hbar^6)$: sixth-order quantum diffusion equation

$$\partial_t u = \operatorname{div}\left(u\nabla\Big[\sum_{i,j=1}^{d}\Big(\frac{1}{2}(\partial_{ij}^2 \log u)^2 + \frac{1}{u}\partial_{ij}^2(u\partial_{ij}^2 \log u)\Big)\Big]\right). \qquad \text{(QD6)}$$

[1D version, Jüngel and Milišić '09.]

Key tool – entropy production inequalities

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   - algorithmic construction of entropies based on systematic treatment of integration by parts formulae

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$$\frac{\mathrm{d}}{\mathrm{d}t} E_1[u(t)] + c \int_{\mathbb{T}^d} \left(\|\nabla^3 \sqrt{u}\|^2 + |\nabla \sqrt[6]{u}|^6 \right) \mathrm{d}x \leq 0, \quad t > 0.$$

[Matthes '10.]

   - global in time existence of solutions, exponential convergence to the homogeneous steady state

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3. Structure preserving numerical scheme for the fourth-order quantum diffusion equation.

Monge - Kantorovich problem

$$\xrightarrow{M}$$

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Zeno of Elea (5. ct. BC.): no solution!

Discussion on a role of PDEs in ACROSS project

Optimal transport problems and Wasserstein distance

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$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \quad \rho(0) = \rho_0, \ \rho(T) = \rho_T$$

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Kinetic energy

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Wasserstein distance

$$W(\rho_0, \rho_T)^2 = \inf_{(\rho,v)} E(\rho, v) = \inf_M \int_{\mathbb{R}^d} |x - M(x)|^2 \rho_0(x) \mathrm{d}x.$$

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Wide range of applications: analysis of PDEs, statistics, image restoration, multisensor multitarget tracking

Discussion on a role of PDEs in ACROSS project

Swarming models

Modelling of the collective behavior of interacting agents: birds (starlings, geese) fish, insects, certain mammals (wildebeasts, sheep), free-flying spacecrafts, ...

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1. Particle models - based on the combination of self-propelling, friction and attraction-repulsion phenomena:

$$\dot{x}_i = v_i,$$
$$\dot{v}_i = (\alpha - \beta|v_i|^2)v_i - \frac{1}{N}\sum_{j\neq i}\nabla U(|x_i - x_j|),$$

for $i = 1, ..., N$, $\alpha, \beta \geq 0$; $U : \mathbb{R}^d \to \mathbb{R}$ – Morse potential given by

$$U(|x|) = -C_A e^{-|x|/l_A} + C_R e^{-|x|/l_R}.$$

2. Kinetic models - Boltzmann equation for the particle density $f(t, x, v)$ (density of agents)

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

$Q(f, f)$ – interaction operator given by

$$Q(f,f)(x,v) = \varepsilon \int_{\mathbb{R}^{2d}} \Big(\frac{1}{J} f(x, v_*) f(y, w_*) - f(x,v) f(y,w) \Big) \, \mathrm{d}w \, \mathrm{d}y.$$

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3. Hydrodynamic models - from the kinetic model with additional assumptions:

$$\begin{aligned}&\partial_t\rho + \mathrm{div}(\rho u) = 0,\\ &\rho\partial_t u_i + \mathrm{div}(\rho u u_i) = \rho(\alpha - \beta|u|^2)u_i - \rho(\partial_{x_i}U * \rho),\end{aligned}$$

where $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$ and $\rho u = \int_{\mathbb{R}^d} f(t,x,v)v\,\mathrm{d}v$.

Thank you for your attention!

Hvala na pažnji!