FIR Fractional Hilbert Transformers With Raised-Cosine Magnitude Response

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Outline

- Introduction
- Fractional Hilbert transformer with raised-cosine magnitude response
- Design of FIR
  - fractional Hilbert transformers
  - conventional Hilbert transformers
  - complex Hilbert filters
- Examples
- Conclusions

Introduction

- Fractional Hilbert transformers are phase shifters
  - 90-degrees phase shifters are called conventional transformers
- The design of fractional transformers often includes the design of conventional transformers
- Some designs of fractional transformers can be used to obtain complex Hilbert filters
- Nowadays, they are usually realized as digital systems

Introduction

- Transfer function of transformers
  - FIR
  - IIR
- Common design methods
  - FIR conventional transformers
    - minimax, least-squares, weighted least-squares, maximally flat and flat approximations
  - FIR fractional transformers
    - from previously designed conventional transformer
    - directly design
      - window method
      - maximally flat approximation

Introduction

- We found the Fourier series method useful for the design of fractional transformers
  - originates from early phase of FIR filter design
  - results in the least-squares approximation
  - the impulse response is truncated by the Fourier window

$\Rightarrow$ oscillatory behavior of the obtained magnitude (the Gibbs phenomenon)

- Decreasing the Gibbs phenomenon
  - window with smooth decay
  - smoothing the transition band

Introduction

- We propose FIR fractional Hilbert transformers with raised-cosine magnitude response
  - good energy concentration of the impulse response
  - good control of the band of interest
  - closed-form method
  - fast and robust design

Fractional Hilbert Transformer With Raised-Cosine Magnitude Response

Frequency Response

- The ideal fractional Hilbert transformer is an ideal phase shifter
- The ideal frequency response

$$H_i(\omega) = \begin{cases} e^{j\varphi} & , -\pi < \omega < 0 \\ e^{-j\varphi} & , \; 0 < \omega < \pi \end{cases}$$

unity magnitude　　constant phase

φ is fractional phase shift

$\Rightarrow$ The ideal response is approximated within some band of interest

- We want to design a transformer using the Fourier series method

  $\Rightarrow$ The impulse response should be well localized

  $\Rightarrow$ Smooth transition bands are required

- We use bandpass raised-cosine magnitude response

$$A_{BP}(\omega) = \begin{cases} \frac{1}{2} - \frac{1}{2}\cos\left(\pi \frac{|\omega| - \omega_1}{\omega_2 - \omega_1}\right) & , \omega_1 < |\omega| < \omega_2 \\ 1 & , \omega_2 \le |\omega| \le \omega_3 \\ \frac{1}{2} + \frac{1}{2}\cos\left(\pi \frac{|\omega| - \omega_3}{\omega_4 - \omega_3}\right) & , \omega_3 < |\omega| < \omega_4 \\ 0 & , \text{otherwise} \end{cases}$$

transition band

band of interest

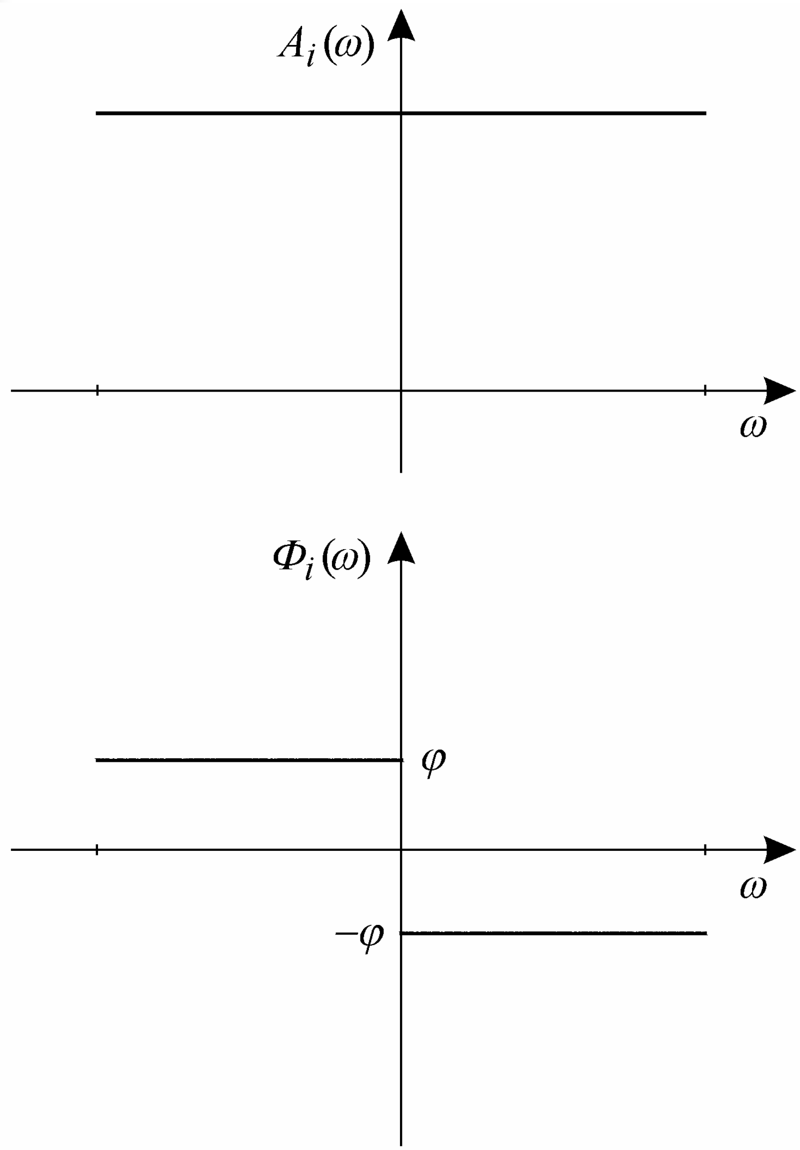
transition band

- The desired frequency response

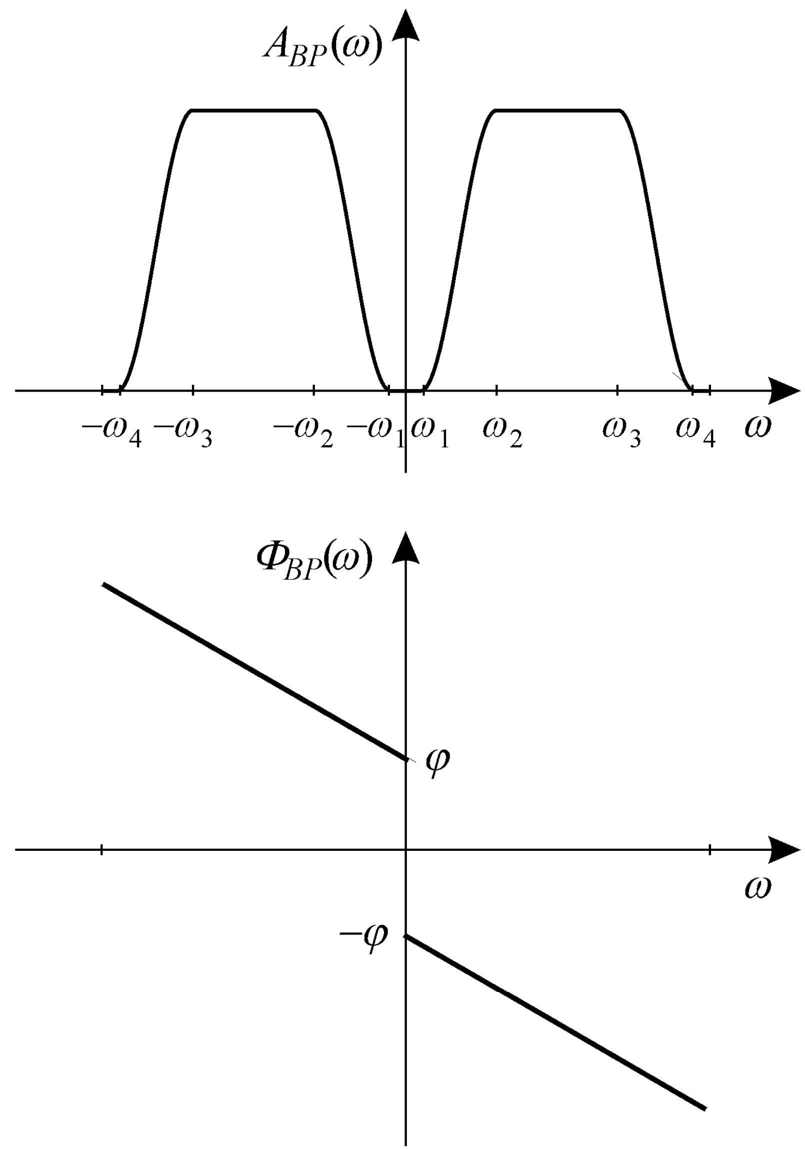
$$H_{BP}(\omega) = \begin{cases} A_{BP}(\omega) e^{j(\varphi - D\omega)} & , -\pi \le \omega < 0 \\ A_{BP}(\omega) e^{-j(\varphi + D\omega)} & , 0 \le \omega < \pi \end{cases}$$

D is delay

Ideal frequency response



Desired frequency response



Impulse Response

- The linear combination of two lowpass responses

$$H_{BP}(\omega) = H_{LP2}(\omega) - H_{LP1}(\omega)$$

$$\omega_3 < |\omega| < \omega_4 \qquad \omega_1 < |\omega| < \omega_2$$

$$H_{LP}(\omega) = \begin{cases} A_{LP}(\omega)e^{j(\varphi - D\omega)} & , -\pi \le \omega < 0 \\ A_{LP}(\omega)e^{-j(\varphi + D\omega)} & , \ 0 \le \omega < \pi \end{cases}$$

$$A_{LP}(\omega) = \begin{cases} 1 & , \ 0 \le |\omega| \le \omega_p \\ \frac{1}{2} + \frac{1}{2}\cos\left(\pi \frac{|\omega| - \omega_p}{\omega_s - \omega_p}\right) & , \ \omega_p < |\omega| < \omega_s \\ 0 & , \ \text{otherwise} \end{cases}$$

- The lowpass impulse response

$$h_{LP}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(\omega)e^{j\omega n} d\omega$$

$$h_{LP}(n)=\begin{cases}\underline{s(n)a(n)+b(n)} & ,\ n\neq D\\ \dfrac{\omega_p+\omega_s}{2\pi}\cos\varphi & ,\ n=D\end{cases}$$

$$s(n)=\frac{\sin\left[\dfrac{\omega_p+\omega_s}{2}(n-D)-\varphi\right]}{\pi(n-D)}$$

$$a(n)=\begin{cases}\dfrac{\cos\left[\dfrac{\omega_s-\omega_p}{2}(n-D)\right]}{1-\left[\dfrac{\omega_s-\omega_p}{\pi}(n-D)\right]^2} & ,\ n\neq D\pm\dfrac{\pi}{\omega_s-\omega_p}\\ \dfrac{\pi}{4} & ,\ n=D\pm\dfrac{\pi}{\omega_s-\omega_p}\end{cases}$$

$$b(n)=\frac{\sin\varphi}{\pi(n-D)}$$

- The bandpass impulse response

$$h_{BP}(n)=h_{LP2}(n)-h_{LP1}(n)$$

$$h_{BP}(n)=\begin{cases}\underline{s_2(n)a_2(n)-s_1(n)a_1(n)} & ,\ n\neq D\\ \dfrac{\omega_3+\omega_4-\omega_1-\omega_2}{2\pi}\cos\varphi & ,\ n=D\end{cases}$$

Design of Fractional Transformers

- The transfer function of an FIR fractional Hilbert transformer of the *N*th order

$$H_{FHT}(z) = \sum_{n=0}^{N} h_{FHT}(n) z^{-n}$$

- Fourier series method

$$h_{FHT}(n) = h_{BP}(n) \quad ; \quad n = 0, 1, \ldots, N$$

$\Rightarrow$ Additional window is not required!

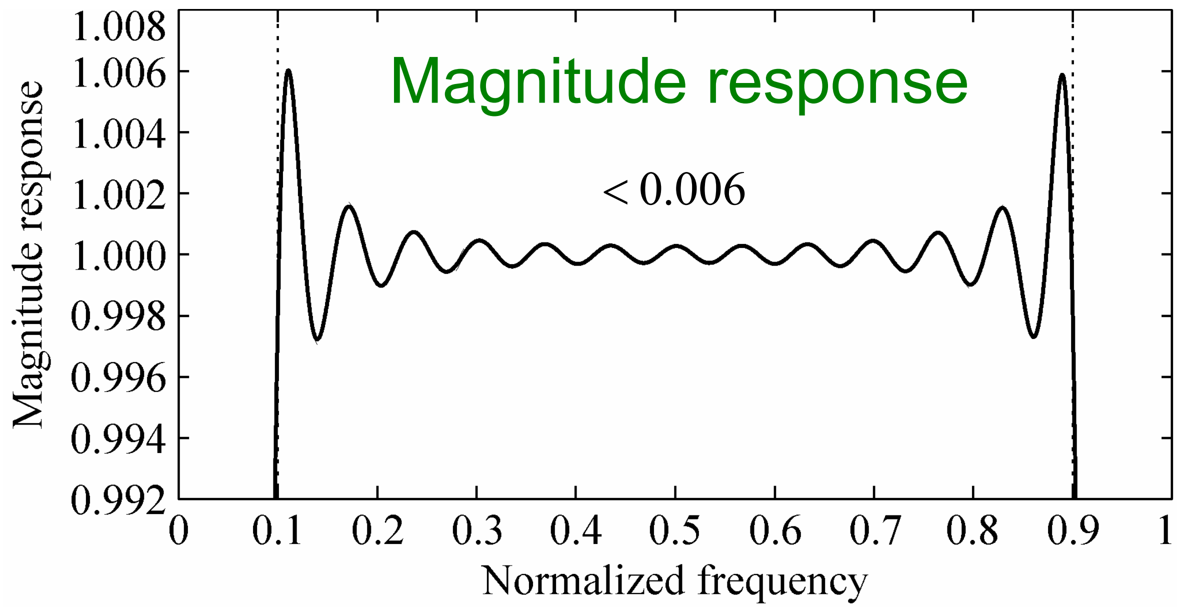
- The best LS approximation is obtained for delay

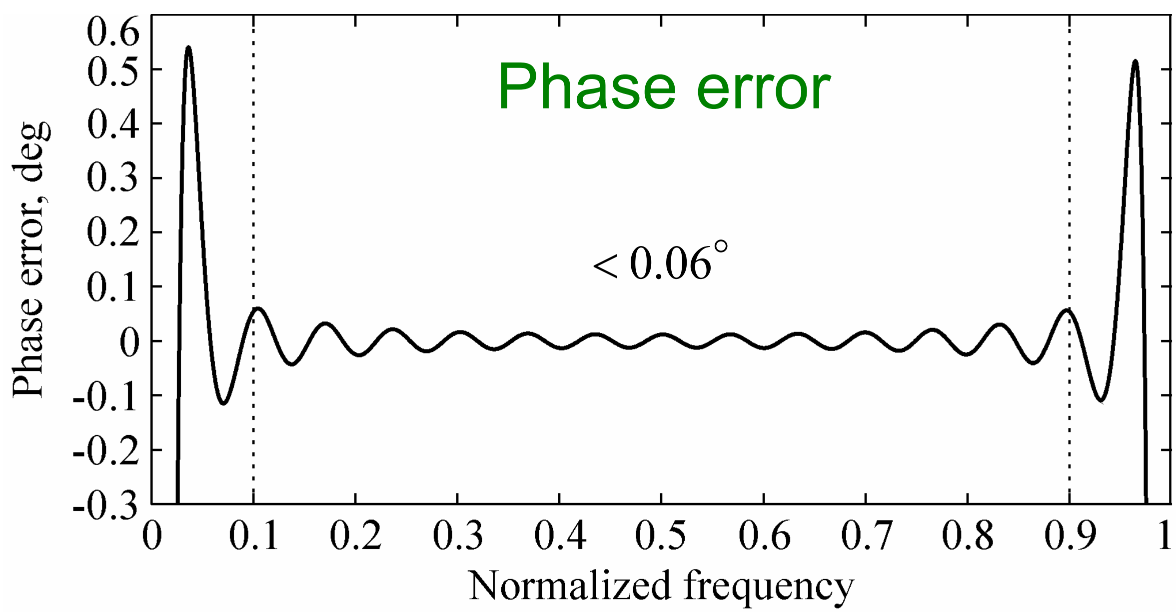
$$D = \frac{N}{2}$$

Design of Fractional Transformers

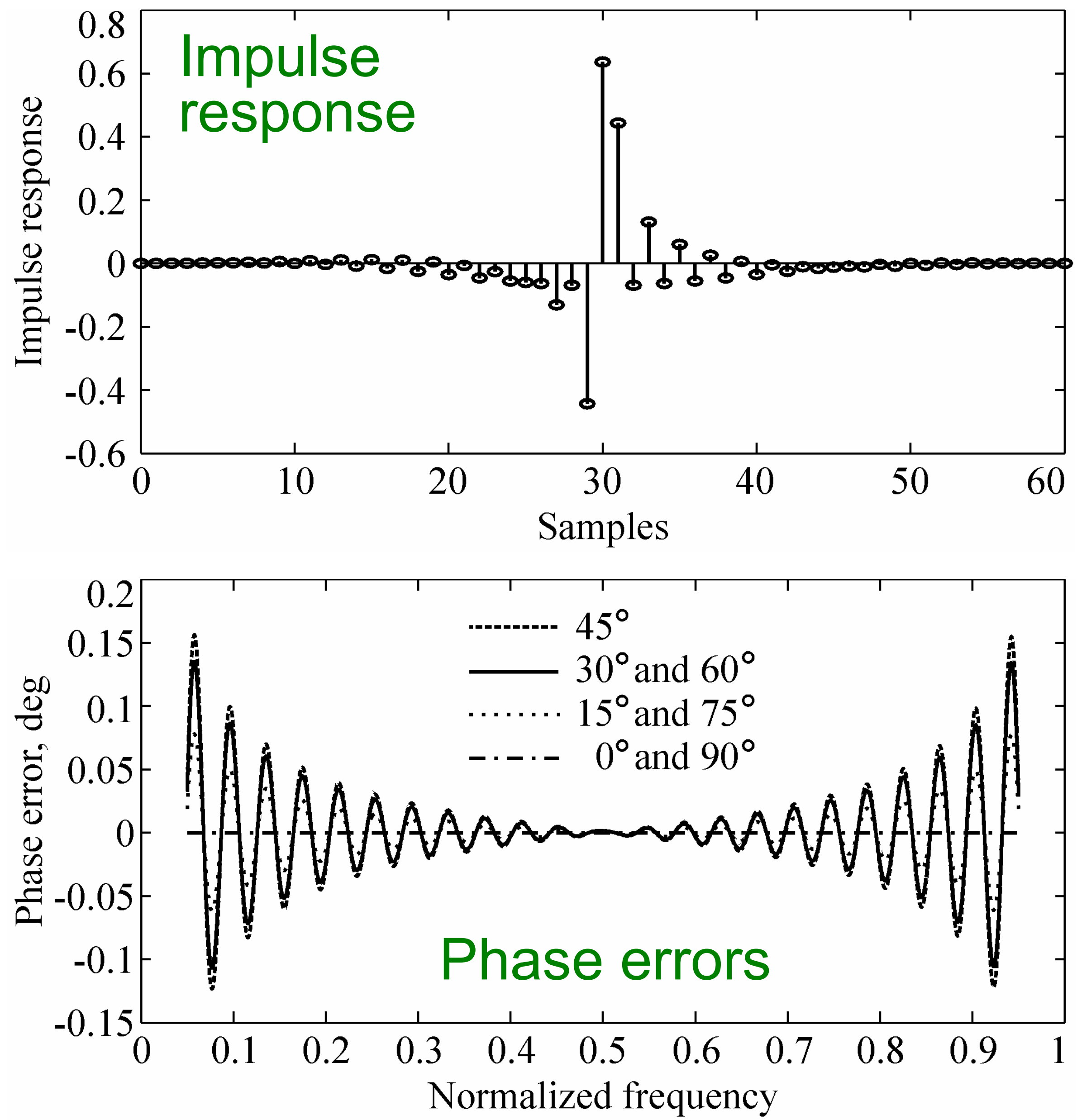
Example 1

- N=60
- $\varphi=\pi/4$
- $\omega_1=0$
  $\omega_2=0.1\pi$
  $\omega_3=0.9\pi$
  $\omega_4=\pi$





Design of Fractional Transformers



Example 2

- N=100
- various φ
- ω_1=0
  ω_2=0.05π
  ω_3=0.95π
  ω_4=π

Design of Fractional Transformers

- The quality of approximation is determined by the energy of the neglected impulse-response tails

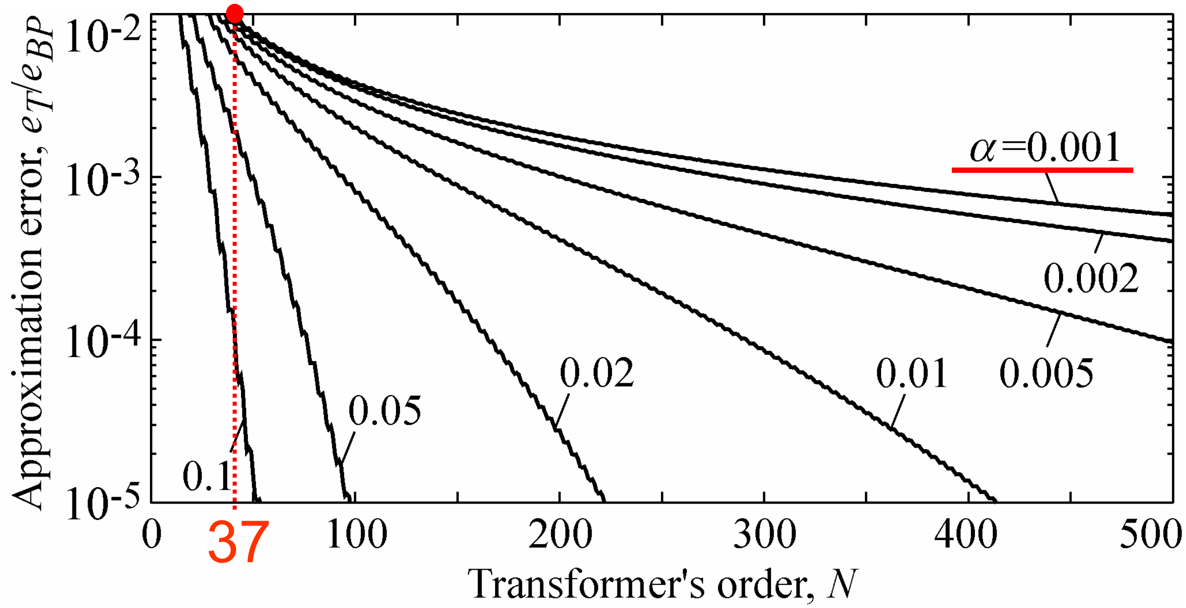
$$e_T = e_{BP} - e_{FHT}$$

$$e_{BP} = \frac{3(\omega_4 - \omega_1) + 5(\omega_3 - \omega_2)}{8\pi} \qquad e_{FHT} = \sum_{n=0}^{N} [h_{FHT}(n)]^2$$

- The relative approximation error = $\dfrac{e_T}{e_{BP}}$

Example 3

- $\varphi = \pi/4$
- $\omega_1 = 0$
  $\omega_2 = \alpha\pi$
  $\omega_3 = (1-\alpha)\pi$
  $\omega_4 = \pi$



Design of Conventional Transformers

- Special case in the design of fractional transformers
  - $\varphi=\pi/2$
- Special case in the design of conventional transformers
  - N is odd
  - $\omega_3=\omega_4=\pi$

$\Rightarrow$ Highpass transformer

- Impulse response of highpass transformer

$$h_{BP}(n)=s_2(n)a_2(n)-s_1(n)a_1(n) \quad , \quad n \neq D$$

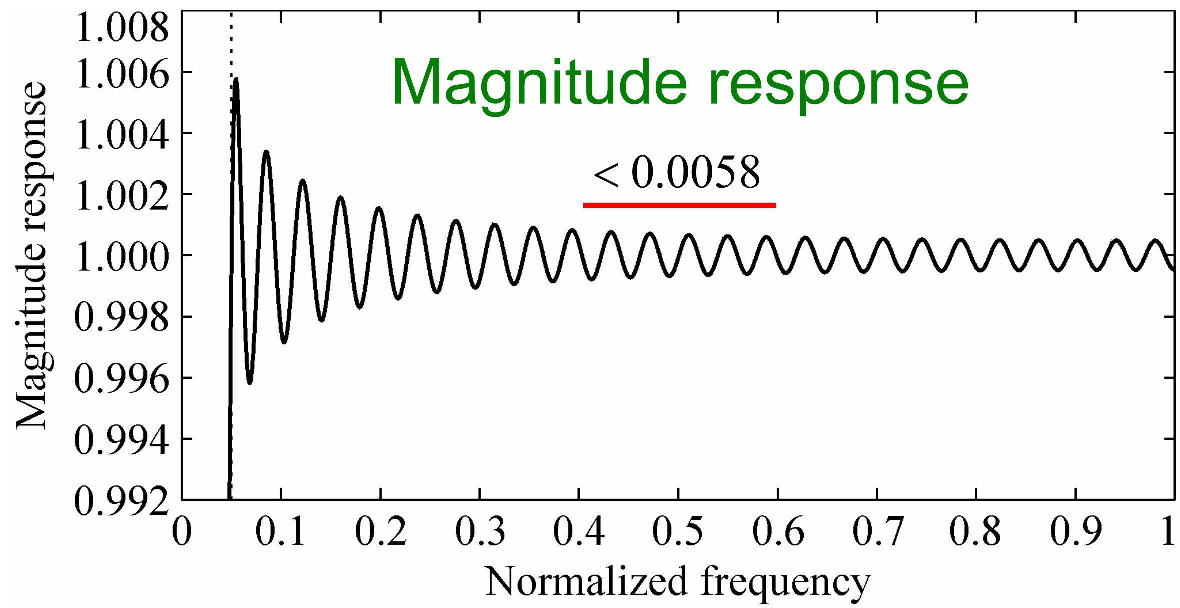
($s_2(n)a_2(n)$) $\Downarrow$ =0

$$\Rightarrow \quad h_{HP}(n)=-s_1(n)a_1(n)$$

Design of Conventional Transformers

Example 4

- N=101
- ω_1=0
  ω_2=0.05π



- The bandpass transformer with N=100, ω_3=0.95π and ω_4=π has the ripple <0.0065

Design of Complex Hilbert Filters

- The presented transformers can be used in the design of systems with complex coefficients

  $\Longrightarrow$ Hilbert filters

- Hilbert filters are bandpass filters that pass only positive frequencies
- We propose the raised-cosine magnitude over the positive frequencies
- Impulse response

$$h_{PP}(n) = \frac{1}{2} h_{BP}(n) + j \frac{1}{2} \hat{h}_{BP}(n)$$

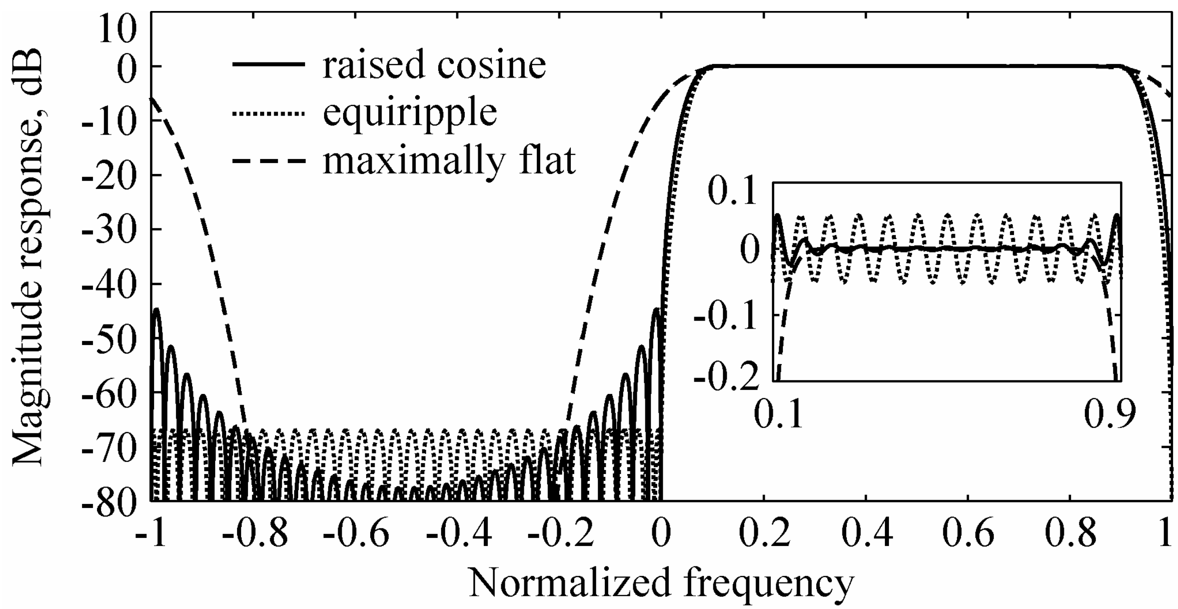
$\varphi = \beta$ $\qquad$ $\varphi = \beta + \pi/2$

β is an arbitrary phase shift

Design of Complex Hilbert Filters

Example 5

- Raised-cosine filter with
  N=61, $\beta=\pi/4$, $\omega_1=0$, $\omega_2=0.1\pi$, $\omega_3=0.9\pi$, $\omega_4=\pi$
- Comparison with
  - equiripple filter with the same passband ripple
  - maximally-flat filter with flatness at $\omega=\pi/2$



Conclusions

- The Fourier series method might still be attractive because it is simple and straightforward
- We used it to develop a class of fractional Hilbert transformers
- The transformers approximate raised-cosine magnitude with fractional phase shift in the least-squares sense
- The used impulse response
  - is well localized in time
  - enables the design of efficient Hilbert transformers and filters without the need for additional window