$\mathcal{L}_p$-Stability with Respect to Sets Applied Towards Self-Triggered Communication for Single-Integrator Consensus

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***Abstract*— In this paper, we formulate and study the concept of $\mathcal{L}_p$-stability with respect to a set. This robustness concept generalizes the standard $\mathcal{L}_p$-stability notion towards control systems designed to steer the system state into the vicinity of a set rather than of a point. We focus on stable LTI systems with the property that all eigenvalues with zero real part are located in the origin. Employing the Real Jordan Form, we devise a mechanism for computing upper bounds associated with $\mathcal{L}_p$-stability and $\mathcal{L}_p$ to $\mathcal{L}_p$ detectability with respect to the equilibrium manifold. Notable examples of this class of LTI systems arise from consensus control. In a self-triggered realization of consensus control problems, each agent broadcasts its state only when necessary in order to achieve consensus. Bringing together $\mathcal{L}_p$-stability with respect to the consensus manifold and the small-gain theorem, we develop self-triggering for single-integrator consensus with fixed and switching network topology. In addition, we show that this consensus problem is Input-to-State Stable with respect to the consensus manifold. Finally, our results are corroborated by numerical simulations.**

I. Introduction

Consider a linear time-invariant (LTI) system

$$\dot{x} = Ax + B\omega, \quad y = Cx + D\omega, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state, $\omega \in \mathbb{R}^{n_\omega}$ denotes the disturbance, and $y \in \mathbb{R}^{n_y}$ denotes the output of the system. According to [1, Corollary 5.2], the following holds: if A is Hurwitz, the system (1) is $\mathcal{L}_p$-stable from ω to y for each $p \in [1, \infty]$. As a result, the class of *asymptotically stable* LTI systems is commonplace and its $\mathcal{L}_p$-stability is frequently addressed and well established [2].

However, a notable class of decentralized control problems, called *consensus* problems, yields *stable* LTI closed-loop systems (in the sense of [3, Theorem 6.3]) with the property that all eigenvalues with zero real part are located in the origin [4]. Consensus problems seek primarily for an agreement; hence, the actual value of the *consensus point* depends on the agents' initial conditions. Another example of this class of stable LTI systems is found in a quadrotor control design [5, Section 3]. The kernel (or null space) of the associated state matrix is nontrivial and spanned by the eigenvectors corresponding to the zero eigenvalue(s). In other words, the kernel represents the *set of equilibrium points*. Having that said, it is more appropriate to consider $\mathcal{L}_p$-stability with respect to this set than with respect to a specific equilibrium point. More motivating examples for analyses of stability with respect to sets can be found in [6]–[8]. For the sake of brevity, we write "w.r.t." instead of "with respect to" in the remainder of the paper.

Self-triggered communication is a variant of the *event-triggered* communication paradigm [9]–[12]. As the name of the latter suggests, information exchange takes place upon the occurrence of significant events, called *triggering events*, related to stability or performance of control systems. Utilizing previously exchanged information, self-triggering predicts the occurrence of triggering events and, based on such predictions, induces an exchange of up-to-date information. Self-triggering aims at reducing requirements posed on sensors and processors in embedded systems.

Works somewhat similar to the one presented herein are reported in [7] and [8]. While [7] and [8] are interested in Input-to-State Stability (ISS), this paper is interested in input-output $\mathcal{L}_p$-stability. Works regarding event- and self-triggering in consensus control are found in [10] and [12], respectively. However, [10] and [12] do not take disturbances nor switching topologies into account. In addition, the authors in [13] tailor several ternary controllers for self-triggered practical consensus and show that those controllers posses some desirable features (e.g., no global information on graph topology is needed, robustness with respect to skews in the agents local clocks, delays, and quantization). As opposed to [13], our methodology, as well as the methodologies in [10] and [12], aims at devising self-triggering for a variety of existing control schemes that are designed on the premise of continuous information flows.

The contributions of this paper are fourfold: a) the formulation of $\mathcal{L}_p$-stability and $\mathcal{L}_p$ to $\mathcal{L}_p$ detectability w.r.t. a set; b) the design of a computing mechanism for constants and gains pertaining to these stability notions; c) the design of stabilizing communication instants for single-integrator consensus problems with fixed and switching network topologies; d) the consideration of disturbances as well as directed and unbalanced topologies. We point out that [10] considers balanced while [12] and [13] consider undirected topologies.

II. Mathematical Preliminaries

A. Notation

To shorten the notation, we use $(x, y) := [x^\top \quad y^\top]^\top$. The dimension of a vector x is denoted n_x. In this paper, $\| \cdot \|$ refers to the Euclidean norm of a vector. If the argument

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of $\|\cdot\|$ is a matrix, then it denotes the induced matrix 2-norm. The set of all eigenvalues of a matrix A is denoted $\lambda(A)$. The kernel of a matrix A is denoted $\mathrm{Ker}(A)$. Given $x \in \mathbb{R}^n$, we define $\overline{x} = (|x_1|, |x_2|, \ldots, |x_n|)$, where $|\cdot|$ denotes the absolute value function. When the argument of $|\cdot|$ is a set, then it denotes the cardinality of the set. Given $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, the partial order $\preceq$ is given as $x \preceq y \iff x_i \leq y_i$, $\forall i \in \{1, \cdots, n\}$. An n-dimensional vector with all entries 0 (respectively, 1) is denoted $\mathbf{0}_n$ (respectively, $\mathbf{1}_n$). Similarly, an $n \times m$ matrix with all zero entries is $\mathbf{0}_{n\times m}$. Likewise, the $n \times n$ identity matrix is denoted I_n. The set $\mathcal{A}_n^+$ denotes the subset of all $n \times n$ matrices that are symmetric and have nonnegative entries. In addition, let $\mathbb{R}_+^n$ denote the nonnegative orthant. We use

$$\|f[a,b]\|_{p,\mathcal{B}} := \left(\int_{[a,b]} \|f(s)\|_{\mathcal{B}}^p \mathrm{d}s \right)^{1/p}, \tag{2}$$

where $\|f(s)\|_{\mathcal{B}} := \inf_{b\in\mathcal{B}} \|f(s) - b\|$, to denote the $\mathcal{L}_p$-norm w.r.t. a set $\mathcal{B} \subset \mathbb{R}^n$ of a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}^n$ restricted to the interval $[a,b] \subset \mathbb{R}$. Lastly, when $\mathcal{B} = \mathbf{0}_n$, we write $\|f[a,b]\|_p$ since this represents the standard $\mathcal{L}_p$-norm.

B. Graph Theory

A *directed graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is a nonempty set of *nodes* (or vertices) with unique ID numbers, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of the corresponding *edges*. When the edge (i,j) belongs to $\mathcal{E}$, it means that there is an information flow from the node i to the node j. We do not allow *self-loops*, i.e., edges that connect a vertex to itself. The set of *neighbors* of the node v_i is $\mathcal{N}_i = \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$ which is all nodes that the node v_i can obtain information from. A *directed tree* is a directed graph in which every node has exactly one parent except for one node. A *subgraph* $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ of $\mathcal{G}$ is a graph such that $\mathcal{V}^s \subseteq \mathcal{V}$ and $\mathcal{E}^s \subseteq \mathcal{E} \cap (\mathcal{V}^s \times \mathcal{V}^s)$. A *directed spanning tree* $\mathcal{G}^s$ of $\mathcal{G}$ is a subgraph of $\mathcal{G}$ such that $\mathcal{G}^s$ is a directed tree and $\mathcal{V}^s = \mathcal{V}$. A graph $\mathcal{G}$ *contains* a directed spanning tree if a directed spanning tree is a subgraph of $\mathcal{G}$.

Given a graph $\mathcal{G}$, the graph Laplacian matrix $L \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ is defined as $L = [l_{ij}]$, where l_{ij} equals -1 for $j \in \mathcal{N}_i$, $|\mathcal{N}_i|$ for $j = i$, and 0 otherwise.

C. Stability Notions

Consider a nonlinear hybrid (or impulsive) system [6]

$$\Sigma \quad \begin{cases} \left.\begin{aligned} \dot{x} &= f(x,\omega) \\ y &= g(x,\omega) \end{aligned}\right\} t \in \bigcup_{i\in\mathbb{N}_0} [t_i, t_{i+1}), \\ x(t^+) = h(x(t)) \qquad t \in \mathcal{T}, \end{cases} \tag{3}$$

where x, y and ω are defined as in (1). We assume enough regularity on f and h to guarantee existence of the solutions given by right-continuous functions $t \mapsto x(t)$ on $[t_0, \infty)$ starting from x_0 at $t = t_0$. Jumps of the state x (or impulses) occur at each $t \in \mathcal{T} := \{t_i : i \in \mathbb{N}\}$. The value of the state after the jump is given by $x(t^+) = \lim_{t' \searrow t} x(t')$ for each $t \in \mathcal{T}$.

In the following definitions, we use the set

$$\mathcal{B}_y := \{y \in \mathbb{R}^{n_y} | \exists b \in \mathcal{B} \text{ such that } y = g(b, \mathbf{0}_{n_\omega})\}, \tag{4}$$

where $\mathcal{B} \subseteq \mathbb{R}^{n_x}$.

Definition 1: (global exponential stability w.r.t. a set) For $\omega \equiv \mathbf{0}_{n_\omega}$, the system Σ is Globally Exponentially Stable (GES) w.r.t. a set $\mathcal{B}$ if there exist $k, l \geq 0$ such that $\|x(t)\|_{\mathcal{B}} \leq k \exp(-l(t - t_0))\|x(t_0)\|_{\mathcal{B}}$ for all $t \geq t_0$ and for any $x(t_0)$.

Definition 2: (Input-to-State Stability w.r.t. a set) The system Σ is Input-to-State Stable (ISS) w.r.t. a set $\mathcal{B}$ if there exist a class-$\mathcal{KL}$ function β and a class-$\mathcal{K}_\infty$ function γ such that, for any $x(t_0)$ and every input ω, the corresponding solution $x(t)$ satisfies $\|x(t)\|_{\mathcal{B}} \leq \beta(\|x(t_0)\|_{\mathcal{B}}, t - t_0) + \gamma(\|\omega[t_0, t]\|_\infty)$.

Definition 3: ($\mathcal{L}_p$-stability w.r.t. a set) Let $p \in [1, \infty]$. The system Σ is $\mathcal{L}_p$-stable from ω to y w.r.t. a set $\mathcal{B}$ with gain $\gamma \geq 0$ if there exists $K \geq 0$ such that $\|y[t_0, t]\|_{p,\mathcal{B}_y} \leq K\|x(t_0)\|_{\mathcal{B}} + \gamma\|\omega[t_0, t]\|_p$ for any $t \geq t_0$, $x(t_0)$ and ω.

Definition 4: ($\mathcal{L}_p$ to $\mathcal{L}_p$ detectability w.r.t. a set) Let $p \in [1, \infty]$. The state x of Σ is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from (ω, y) to x w.r.t. a set $\mathcal{B}$ with gain $\gamma_d \geq 0$ if there exists $K_d \geq 0$ such that $\|x[t_0, t]\|_{p,\mathcal{B}} \leq K_d\|x(t_0)\|_{\mathcal{B}} + \gamma_d\|y[t_0, t]\|_{p,\mathcal{B}_y} + \gamma_d\|\omega[t_0, t]\|_p$ for any $t \geq t_0$, $x(t_0)$ and ω.

Definitions 1 and 2 are motivated by [1], [7] and [8], while Definitions 3 and 4 are motivated by [14]. Notice that K, γ, K_d and γ_d in Definitions 3 and 4 are not unique.

D. Switched Systems and Average Dwell-Time

Consider a family of systems (3) indexed by the parameter ρ taking values in a set $\mathcal{P} = \{1, 2, \ldots, m\}$. Let us define a right-continuous and piecewise constant function $\sigma : [t_0, \infty) \to \mathcal{P}$ called a *switching signal*. The role of σ is to specify which system is active at any time $t \geq t_0$. The resulting switched system investigated herein is given by

$$\Sigma_\sigma \quad \begin{cases} \left.\begin{aligned} \dot{x} &= f_\sigma(x,\omega) \\ y &= g(x,\omega) \end{aligned}\right\} t \in \bigcup_{i\in\mathbb{N}_0} [t_i, t_{i+1}), \\ x(t^+) = h_\sigma(x(t)) \qquad t \in \mathcal{T}. \end{cases} \tag{5}$$

For each switching signal σ and each $t \geq t_0$, let $N_\sigma(t, t_0)$ denote the number of discontinuities, called *switching times*, of σ on the open interval (t_0, t). We say that σ has *average dwell-time* τ_a [15] if there exist two positive numbers N_0 and τ_a such that $N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_a}$ for every $t \geq t_0$. In this paper, different values of σ correspond to different topologies L, while state jump instants t_i's indicate when an exchange of information takes place.

III. Problem Statement and Assumptions

Consider an LTI system (1) satisfying the following:

Assumption 1: All eigenvalues of A have nonpositive real parts. In addition, the eigenvalues with zero real part are located in the origin.

From [3, Definition B.14] we deduce the following. The algebraic multiplicity of A, denoted $\mathscr{A}$, equals the multiplicity of zero as a root of the characteristic polynomial. The dimension of the kernel of A equals the geometric multiplicity of the zero eigenvalue and is denoted $\mathscr{G}$.

Assumption 2: A is such that $\mathscr{A} = \mathscr{G}$.

Now, take $\mathcal{B}$ to be the kernel of A, i.e.,

$$\mathcal{B} = \mathrm{Ker}(A). \tag{6}$$

Apparently, $\|\cdot\|_{\mathcal{B}}$ measures how close the system state x is from the nearest equilibrium point, i.e., from the *equilibrium manifold*. When applied to single-integrator consensus control, $\|\cdot\|_{\mathcal{B}}$ measures how close the agents are from achieving consensus, i.e., from the *consensus manifold*. Let us now state the problems solved in Sections IV and V, respectively.

Problem 1: Provided that an LTI system (1) satisfies Assumptions 1 and 2, establish $\mathcal{L}_p$-stability from ω to y w.r.t. $\mathcal{B}$ given by (6) in the sense of Definition 3. In addition, develop a mechanism to compute a suitable constant K and gain γ. Under the same assumptions, establish $\mathcal{L}_p$ to $\mathcal{L}_p$ detectability of x from (ω, y) w.r.t. $\mathcal{B}$ given by (6) in the sense of Definition 4. In addition, develop a mechanism to compute a suitable constant K_d and gain γ_d.

Problem 2: Accommodate the small-gain theorem towards $\mathcal{L}_p$-stability w.r.t. a set, and develop self-triggering for the single-integrator consensus problem with fixed and time-varying communication topologies.

Single-integrator consensus yields closed-loop dynamics that satisfy Assumptions 1 and 2. However, our framework is not constrained only to such a specific problem.

IV. $\mathcal{L}_p$-STABILITY WITH RESPECT TO SET

The approach presented in this section originates from the following observations. Notice that $\|\cdot\|_{\mathcal{B}}$ in (2) is a seminorm when $\mathcal{B} \neq \mathbf{0}_{n_x}$. Thus, $\|\cdot\|_{\mathcal{B}}$ does not separate points but, rather, equivalence classes when $\mathcal{B}$ is not a singleton. In case $\mathcal{B}$ is a subspace of $\mathbb{R}^{n_x}$, $\|\cdot\|_{\mathcal{B}}$ is a norm for the normed space $\mathbb{R}^{n_x}/\mathcal{B}$. The space $\mathbb{R}^{n_x}/\mathcal{B}$ is called the quotient space. Due to (6), the set $\mathcal{B}$ considered herein is a closed subspace of $\mathbb{R}^{n_x}$. Consequently, $\mathcal{B}$ is a manifold. In quotient spaces defined in this manner, the following holds:

$$\|x + b\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}, \tag{7}$$

for all $b \in \mathcal{B}$ and $x \in \mathbb{R}^{n_x}$. In addition, when $\mathcal{B}_y$ is obtained from $\mathcal{B}$ via a linear mapping C, the set $\mathcal{B}_y$ is a subspace of $\mathbb{R}^{n_y}$ and we can restrict ourselves to $\mathbb{R}^{n_y}/\mathcal{B}_y$.

A. Real Jordan Form

Let us introduce a substitution (i.e., change of coordinates)

$$z = Tx, \tag{8}$$

where T is an invertible matrix (i.e., diffeomorphism), such that (1) becomes:

$$\dot{z} = \underbrace{TAT^{-1}}_{A_D} z + TB\omega, \qquad y = CT^{-1}z + D\omega, \tag{9}$$

where

$$A_D = \begin{bmatrix} J_{\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_0 \end{bmatrix}, \tag{10}$$

and J_{λ_i}'s are elementary Jordan blocks of the Real Jordan Form (refer to [16, Section 1.8]). Elementary blocks J_{λ_i}'s are either of the form

$$\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_i & 1 \\ 0 & \dots & \dots & 0 & \lambda_i \end{bmatrix},$$

for a real eigenvalue $\lambda_i \in \lambda(A)$, or of the form

$$\begin{bmatrix} F & I_2 & \mathbf{0}_{2\times 2} & \dots & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & F & I_2 & \dots & \mathbf{0}_{2\times 2} \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \dots & \dots & F & I_2 \\ \mathbf{0}_{2\times 2} & \dots & \dots & \mathbf{0}_{2\times 2} & F \end{bmatrix},$$

where $F = \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$, for a complex eigenvalue $\lambda_i = a + jb \in \lambda(A)$. Notice that $n_z = n_x$.

Recall that the eigenvectors corresponding to complex conjugate eigenvalues are themselves complex conjugate so that the potential calculations involve working in complex spaces. The Real Jordan Form allows us to stay in the real number system by properly choosing a basis of generalized eigenvectors of A. An algorithm for how to choose such a basis is provided in, for example, [16, Section 1.8]. The generalized eigenvectors, that form this basis, have real entries and constitute columns of T^{-1}. Likewise, the entries of T are real. Consequently, TAT^{-1}, TB and CT^{-1} in (9) are matrices over $\mathbb{R}$. In addition, notice that $\mathcal{B}$ is spanned by the last $\mathcal{A}$ columns of T^{-1}. Since T^{-1} is invertible, the first $n_x - \mathcal{A}$ columns of T^{-1} span a complementary space of $\mathcal{B}$, denoted $\mathcal{B}^c$.

B. Reduced System

From the form of A_D given by (10), we infer that the first $n_x - \mathcal{A}$ and the last $\mathcal{A}$ components of z are decoupled. Furthermore, the last $\mathcal{A}$ components of z are in the null space of A_D, i.e., their values are irrelevant. Therefore, we prune the last $\mathcal{A}$ components of z, obtaining z_r, and reduce (9) to

$$\dot{z}_r = A_r z_r + B_r \omega, \qquad y_r = C_r z_r + D_r \omega, \tag{11}$$

where A_r is obtained from A_D by removing the last $\mathcal{A}$ rows and columns of A_D (those contain all zeros anyway), B_r is obtained from TB by removing the last $\mathcal{A}$ rows of TB, and C_r is obtained from CT^{-1} by removing the last $\mathcal{A}$ columns of CT^{-1}. Even though D from (9) remains unaltered, we cast it as D_r in order to retain uniform nomenclature. Notice that the mapping from z into z_r is not a diffeomorphism, i.e., (9) and (11) are not "equivalent". This can be seen from the fact that y_r carries less information than y, i.e.,

$$y = y_r + b_y, \tag{12}$$

where $b_y \in \mathcal{B}_y$ and $\mathcal{B}_y$ is given by (4). The term b_y in (12) is related to the loss of information due to reducing TB to B_r and CT^{-1} to C_r. Basically, the nonzero terms $B'\omega$ and $C'z'$, where B' denotes the last $\mathcal{A}$ rows of TB, C' denotes the last $\mathcal{A}$ columns of CT^{-1} and z' denotes the last $\mathcal{A}$ components of z, are pruned from the system (9) while obtaining (11). The contributions of $B'\omega$ and $C'z'$ towards y lie in $\mathcal{B}_y$ and are denoted b_y.

Notice that A_r is Hurwitz due to its construction. Hence, (11) is $\mathcal{L}_p$-stable from ω to y_r for each $p \in [1, \infty]$. Using [1, Corollary 5.2 & Theorem 5.4], one obtains some K_r and γ_r for the standard $\mathcal{L}_p$-stability of (11).

C. Establishing $\mathcal{L}_p$-stability w.r.t. $\mathcal{B}$

What we have so far is the following:

$$\|y_r[t_0, t]\|_p \leq K_r \|z_r(t_0)\| + \gamma_r \|\omega[t_0, t]\|_p,$$

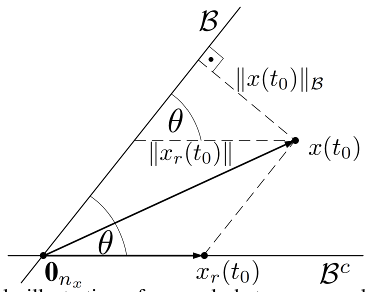

Fig. 1. A simple illustration of an angle between complementary spaces and how it relates to the norms used in our computations.

for any $t \geq t_0$, $z_r(t_0)$ and ω. Using the analogue of (7) in the quotient space $\mathbb{R}^{n_y}/\mathcal{B}_y$, from (12) we infer that $\|y\|_{\mathcal{B}_y} \leq \|y_r\|$. Therefore,

$$\|y[t_0,t]\|_{p,\mathcal{B}_y} \leq K_r\|z_r(t_0)\| + \gamma_r\|\omega[t_0,t]\|_p,$$

for any $t \geq t_0$, $z_t(t_0)$ and ω. Utilizing results from [17], we proceed as follows:

$$\begin{aligned}\|y[t_0,t]\|_{p,\mathcal{B}_y} &\leq K_r\|z_r(t_0)\| + \gamma_r\|\omega[t_0,t]\|_p = \\ &= K_r\|(z_r(t_0), \mathbf{0}_{\mathcal{A}})\| + \gamma_r\|\omega[t_0,t]\|_p = \\ &= K_r\|Tx_r(t_0)\| + \gamma_r\|\omega[t_0,t]\|_p \leq \\ &\leq K_r\|T\|\|x_r(t_0)\| + \gamma_r\|\omega[t_0,t]\|_p \leq \\ &\leq K_r\|T\|\|P\|\|x(t_0)\|_{\mathcal{B}} + \gamma_r\|\omega[t_0,t]\|_p, \end{aligned} \tag{13}$$

where $x_r(t_0)$ is the oblique projection of $x(t_0)$ onto $\mathcal{B}^c$ along (or parallel to) $\mathcal{B}$, P is the oblique projector onto $\mathcal{B}^c$ along $\mathcal{B}$. In general, the following holds $\|P\| = \frac{1}{\sin\theta_{\min}}$, where $\theta_{\min} \in [0, \pi/2]$ satisfies

$$\cos\theta = \max_{\substack{u\in\mathcal{B}, v\in\mathcal{B}^c \\ \|u\|=\|v\|=1}} v^\top u,$$

and represents the minimal angle between $\mathcal{B}$ and $\mathcal{B}^c$ (consult [17] for more details). In our settings, $\theta_{\min} \neq 0$ since $\mathcal{B}$ and $\mathcal{B}^c$ are complementary spaces. Consequently, $\|P\|$ is well defined for our problem of interest. A simple 2D illustration of the relation among θ, $\|x_r(t_0)\|$ and $\|x(t_0)\|_{\mathcal{B}}$ is provided in Figure 1. Obviously, $\|x_r(t_0)\| = \frac{\|x(t_0)\|_{\mathcal{B}}}{\sin\theta}$.

Theorem 1: If an LTI system (1) satisfies Assumptions 1 and 2, then it is $\mathcal{L}_p$-stable from ω to y w.r.t. $\mathcal{B}$ given by (6) for any $p \in [1, \infty]$ in the sense of Definition 3. A suitable choice for K and γ in Definition 3 is as follows:

$$K = K_r\|T\|\|P\|, \qquad \gamma = \gamma_r, \tag{14}$$

where K_r and γ_r pertain to $\mathcal{L}_p$-stability of (11) from ω to y_r.

D. Establishing $\mathcal{L}_p$ to $\mathcal{L}_p$ Detectability w.r.t. $\mathcal{B}$

Let us choose $C = I_{n_x}$ and $D = \mathbf{0}_{n_x \times n_\omega}$ in (1). In light of Theorem 1, one obtains

$$\|x[t_0,t]\|_{p,\mathcal{B}} \leq K_d\|x(t_0)\|_{\mathcal{B}} + \gamma_d\|\omega[t_0,t]\|_p, \tag{15}$$

where K_d and γ_d are obtained in the same manner as K and γ in (14), respectively. Now, we add the nonnegative element $\gamma_d\|y[t_0,t]\|_{p,\mathcal{B}_y}$ to the right hand side of (15) and obtain

$$\begin{aligned}\|x[t_0,t]\|_{p,\mathcal{B}} \leq K_d\|x(t_0)\|_{\mathcal{B}} + \gamma_d\|y[t_0,t]\|_{p,\mathcal{B}_y} + \\ + \gamma_d\|\omega[t_0,t]\|_p. \end{aligned} \tag{16}$$

According to Definition 4, the state x of (1) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from (ω, y) w.r.t. $\mathcal{B}$ given by (6) for any $p \in [1, \infty]$. Notice that Problem 1 is solved now.

Next, we provide a result utilized in Section V.

Proposition 1: $\mathcal{L}_p$-stability from ω to y w.r.t. $\mathcal{B}$ and $\mathcal{L}_p$ to $\mathcal{L}_p$ detectability w.r.t. $\mathcal{B}$ imply $\mathcal{L}_p$-stability from ω to x w.r.t. $\mathcal{B}$.

V. Single-Integrator Consensus

Consider N identical agents given by

$$\dot{\xi}_i = u_i, \qquad \zeta_i = \xi_i, \tag{17}$$

where $\xi_i, \zeta_i, u_i \in \mathbb{R}^{n_\xi}$ are the state, output and control input, respectively, of the i^{th} agent, $i \in \{1, 2, \dots, N\}$. Motivated by [4, Chapter 2], we consider the following decentralized control law

$$u_i = -\sum_{j\in\mathcal{N}_i}(\zeta_i - \zeta_j) + \omega_i. \tag{18}$$

Next, we define the following stack vectors $x := (\xi_1, \xi_2, \dots, \xi_N)$, $y := (\zeta_1, \zeta_2, \dots, \zeta_N)$, and $\omega := (\omega_1, \omega_2, \dots, \omega_N)$. Given the control law (18), the closed-loop dynamic equation (17) becomes

$$\dot{x} = -(L \otimes I_{n_\xi})x + \omega, \qquad y = x, \tag{19}$$

where $\otimes$ denotes the Kronecker product.

Definition 5: Suppose we have a system of N agents given by (17). We say that the agents achieve consensus if $\|y_i - y_j\| \to 0$ as $t \to \infty$ for all $i, j \in \{1, 2, \dots, N\}$.

Let us now assume that $\mathcal{G}$ contains a directed spanning tree. From [4, Lemma 2.4.], we infer that $-L$ satisfies Assumptions 1 and 2. Furthermore, $-L$ has exactly one zero eigenvalue and the associated eigenvector is $\mathbf{1}_{n_\xi}$. From [18, Theorem 13.12.], we infer that algebraic multiplicities of the eigenvalues of $-(L \otimes I_{n_\xi})$ are the respective algebraic multiplicities of the eigenvalues of $-L$ multiplied by n_ξ. In other words, $\mathcal{A} = n_\xi$ for system (19). From [18, Corollary 13.11.], we infer that $\mathcal{G} = n_\xi$ for system (19). Thus, $-(L \otimes I_{n_\xi})$ satisfies Assumptions 1 and 2 as well. In addition, from the second part of [18, Theorem 13.12.] we conclude that $\mathrm{Ker}(-(L \otimes I_{n_\xi}))$ is spanned by n_ξ vectors $\{\mathbf{1}_{n_\xi} \otimes (1, 0, \dots, 0), \mathbf{1}_{n_\xi} \otimes (0, 1, 0, \dots, 0), \dots, \mathbf{1}_{n_\xi} \otimes (0, \dots, 1)\}$ of dimension n_x.

Since we do not consider continuous feedback in (18), the control signal becomes

$$u_i = -\sum_{j\in\mathcal{N}_i}(\hat{\zeta}_i - \hat{\zeta}_j) + \omega_i, \tag{20}$$

where $\hat{\zeta}_j$ is the most recently transmitted value of the output of the j^{th} agent. Let $\mathcal{T}_i := \{t_i^j : j \in \mathbb{N}\}$ denote the set of broadcasting time instants of the i^{th} agent and $\mathcal{T} := \cup_{i=1}^N \mathcal{T}_i$. Next, we introduce the output error vector

$$e = (e_1, \dots, e_N) := (\hat{\zeta}_1 - \zeta_1, \dots, \hat{\zeta}_N - \zeta_N) = \hat{y} - y.$$

The above expression uses $\hat{y} := (\hat{\zeta}_1, \hat{\zeta}_2, \dots, \hat{\zeta}_N)$. Taking e into account, the closed-loop dynamics (19) become

$$\dot{x} = -(L \otimes I_{n_\xi})x - (L \otimes I_{n_\xi})e + \omega. \tag{21}$$

Since $\dot{\hat{y}} = 0$, the corresponding output error dynamics are

$$\dot{e} = -\dot{x}. \tag{22}$$

In what follows, we design the sets of broadcasting (i.e., communication) instants $\mathcal{T}_i$, $i \in \{1, 2, \ldots, N\}$, such that the agents reach consensus in the sense of Definition 5.

A. Designing Broadcasting Instants

Based on the underlying communication topology and information exchanged in the most recent broadcasting instant, our self-triggering mechanism computes when the next exchange of information should take place in order to reach consensus. The time elapsed before the next broadcasting instant is denoted τ. The impact of broadcasting agents' states is as follows: if the i^{th} agent broadcasts at time t, the corresponding components of e reset to zero while other components remain unchanged, i.e.,

$$e_i^+(t) = \mathbf{0}_{n_\varsigma}, \qquad e_j^+(t) = e_j(t), \tag{23}$$

for all $j \in \{1, \ldots, N\}$ such that $j \neq i$. In what follows, we assume that broadcasting instants of the agents coincide for the sake of simplicity. Thus, τ represents the interbroadcasting interval of each agent, i.e., τ is the same for all agents. If one is concerned with message collisions or synchronicity of the broadcasting instants, the scheduling protocols from [19] should be employed.

Now, let us interconnect dynamics (21) and (22) in order to employ the small-gain theorem [1]. To this end, we upper bound the output error dynamics (22) as follows:

$$\bar{\dot{e}} = \overline{(L \otimes I_{n_\xi})x + (L \otimes I_{n_\xi})e - \omega} \preceq A^* \bar{e} + \tilde{y}(x, \omega), \tag{24}$$

where

$$A^* = [a^*_{ij}] := \max\{|c^*_{ij}|, |c^*_{ji}|\}, \tag{25}$$

$$\tilde{y}(x, \omega) := \overline{(L \otimes I_{n_\xi})x - \omega}. \tag{26}$$

In (25), we use $L \otimes I_{n_\xi} = [c^*_{ij}]$. Notice that $A^* \in \mathcal{A}^+_{n_e}$ and $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\omega} \to \mathbb{R}^{n_e}_+$ is a continuous function. With this choice of A^* and $\tilde{y}$, the upper bound (24) holds for all $(x, e, \omega) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_\omega}$ and all $t \in \mathbb{R}$.

Theorem 2: Suppose that $\tau \in (0, \tau^*)$, where $\tau^* := \frac{\ln(2)}{\|A^*\|}$. Then, the output error system (22) is $\mathcal{L}_p$-stable from $\tilde{y}$ to e for any $p \in [1, \infty]$ with a gain

$$\gamma_e = \frac{\exp(\|A^*\|\tau) - 1}{\|A^*\|(2 - \exp(\|A^*\|\tau))}. \tag{27}$$

Next, take (ω, e) to be the input and $\tilde{y}$, obtained in Theorem 2, to be the output of the dynamics (21). We point out that $\tilde{y}$ is an auxiliary signal used to interconnect (21) and (22), but does not exist physically. Notice that $\tilde{y}$ is not linear in x and ω as required in Theorem 1.

Proposition 2: Assume that system (21) with the input (ω, e) and output $y^\dagger := (L \otimes I_{n_\xi})x - \omega$ is $\mathcal{L}_p$-stable w.r.t $\mathcal{B}$ with some constant K and gain γ. Then, the system (21) with the input (ω, e) and output $\tilde{y}$ given by (26) is $\mathcal{L}_p$-stable w.r.t $\mathcal{B}$ with the same constant K and gain γ.

Now, due to Theorem 1 and Proposition 2, this LTI system is $\mathcal{L}_p$-stable w.r.t. $\mathcal{B} = \text{Ker}(-(L \otimes I_{n_\xi}))$ from (ω, e) to $\tilde{y}$ for any $p \in [1, \infty]$. A corresponding gain, obtained via (14) for example, is denoted γ. Notice that systems (21) and (22) are interconnected as depicted in Figure 2.

Theorem 3: If the interbroadcasting interval τ in (27) is such that $\gamma\gamma_e < 1$, then the single-integrator consensus problem is $\mathcal{L}_p$-stable from ω to (x, e) w.r.t. $(\mathcal{B}, \mathbf{0}_{n_e})$ for given $p \in [1, \infty]$.

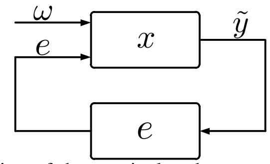

Fig. 2. Interconnection of the nominal and output error dynamics. Jumps of e occur when $t \in \mathcal{T}$.

Remark 1: Notice that $\gamma_e(\tau)$ in (27) is a monotonically increasing function of $\tau \in [0, \tau^*)$. In addition, notice that $\gamma_e(0) = 0$. Due to Theorem 1, we know that $\gamma < \infty$. Since our goal is to design τ such that $\gamma\gamma_e(\tau) < 1$, we first find τ' such that $\gamma\gamma_e(\tau') = 1$, and then compute $\tau = \kappa\tau'$, where $\kappa \in (0, 1)$. Due to monotonicity of $\gamma_e(\tau)$, the obtained τ' is strictly positive; hence, $\tau = \kappa\tau'$ is strictly positive. Consequently, the unwanted Zeno behavior [6] is avoided.

B. Switching Communication Topologies

Since the number of agents is N, there exists only a finite number, say m, of topologies that contain a directed spanning tree. Herein, such topologies are called admissible topologies. From the discussion below Definition 5, we conclude that $\text{Ker}(-(L \otimes I_{n_\xi}))$ is time invariant even though the topologies may be time-varying. Now, one can consider each of m associated reduced systems (11) and conclude that these individual systems are GES uniformly in t_0. Applying the methodology of [19, Section V] to these individual systems and merging the obtained results with the results derived herein, we obtain the following:

Theorem 4: Consider the family of m admissible topologies with the self-triggered broadcasting from Subsection V-A implemented. Then there exists a constant τ_a such that single-integrator consensus problems are ISS from ω to (x, e) w.r.t. $(\mathcal{B}, \mathbf{0}_{n_e})$ for every switching signal σ with average dwell-time τ_a.

In a sense, self-triggering combines attributes of both the time- and event-triggered control paradigm [9]. As suggested by [9], combinations of these two paradigms are often beneficial. Basically, upon arrival of up-to-date information, control signals are updated, the subsequent broadcasting instant is computed in order to preclude destabilizing events (a feature of event-triggering), and the update mechanism commits to the computed subsequent broadcasting instants (a feature of time-triggering). Since papers on self-triggering typically assume state feedback, the subsequent broadcasting instant is routinely computed based on up-to-date state measurements [12], [13]. This type of self-triggering is termed state-triggering. However, the subsequent broadcasting instant can be determined upon both the state and input information [20] or upon promises made by agents [21] as well. Even though the latter update scheme is labeled team-triggering, both of these update schemes are variants of self-triggering.

In the self-triggered update scheme presented herein, the computation of τ via Theorem 3 precludes the small-gain condition from being compromised (refer to [11] for more about this triggering event) and the update scheme commits to the computed τ. Because this paper investigates LTI agents (17) and control laws (18), the upper bound (24) holds globally. Consequently, γ_e in (27) does not depend on (x, e, ω) for any $t \in \mathbb{R}$. Hence, in case the topology is fixed, the rule $\tau = \kappa\tau'$ yields periodic broadcasting. Consideration of nonlinear (or even time-varying) agents or control laws would yield state-triggering, i.e., a state-dependent τ, for a fixed topology (refer to [11]). However, this consideration also entails calculation of constants and gains related to $\mathcal{L}_p$-stability w.r.t. sets of nonlinear (i.e., time-varying) systems which has yet to be devised. A pursuit for a compelling nonlinear example is high on our research agenda.

Due to the commitment of self-triggering to a computed τ, agents know when they should "hear" from their neighbors. The agents exploit this knowledge to detect changes in the communication topology, induce a topology discovery algorithm [19], and compute the respective τ. Apparently, τ adapts to changes in topology based on the (absence of) state information from neighbors as demonstrated in Section VI. As a result, for LTI settings, one could label our self-triggered control scheme as topology-triggering.

VI. Numerical Results

Consider five agents given by (17) and choose $n_\xi = 1$. In addition, consider the following two topologies

$$L_1 = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

Let us compute τ_1 for the topology given by L_1. First, we compute T_1 in (8) obtaining

$$T_1 = \begin{bmatrix} 1.0925 & 2.052 & -0.185 & 0.25 & 1 \\ -1.6624 & 0.5623 & 0.3247 & 0.25 & 1 \\ -0.2151 & -1.3071 & -0.5698 & -0.5 & 1 \\ 1.0925 & 2.052 & -0.185 & -0.5 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The corresponding matrix $A_{D,1}$ is given by

$$A_{D,1} = \begin{bmatrix} -1.1226 & 0.7449 & 0 & 0 & 0 \\ -0.7449 & -1.1226 & 0 & 0 & 0 \\ 0 & 0 & -2.7549 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us consider the case $p = 2$ and apply [1, Theorem 5.4] to the associated reduced system (11). We obtain a gain $\gamma_1 = 3.241$ from (ω, e) to $\tilde{y}$ w.r.t. $\mathcal{B}$. According to (25), we obtain

$$A_1^* = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}, \qquad \|A_1^*\| = 4.1578.$$

The corresponding $\gamma_{e,1}$ as a function of τ_1 is given by (27). By requiring that $\gamma_1\gamma_{e,1} < 1$, we obtain $\tau_1 = 0.107$ s which corresponds to the broadcasting frequency of 9.33 Hz for each agent. By repeating the same steps for L_2, we obtain $\gamma_2 = 2.845$, $\|A_2^*\| = 3.3028$ and $\tau_2 = 0.13$ s which corresponds to the broadcasting frequency of 7.68 Hz for each agent.

In order to verify Theorem 4, we toggle between topologies L_1 and L_2. Numerical results for an illustrative scenario are provided in Figure 3. In this scenario, we choose $\omega_i(t) = \mathcal{N}(0,1)t_{[0,10)} + \mathcal{N}(0,6)t_{[10,20)} + 0t_{[20,30]}$, $i \in \{1, \ldots, N\}$, where $t_\mathcal{I}$ is the indicator function on an interval $\mathcal{I}$. In other words, $t_\mathcal{I} = t$ when $t \in \mathcal{I}$ and zero otherwise. In addition, $\mathcal{N}(a, b)$ denotes the normal distribution with mean a and standard deviation b.

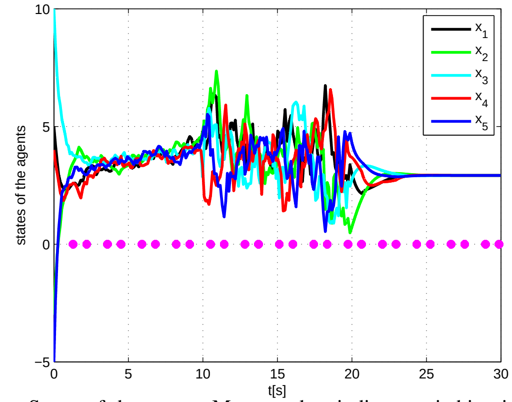

Fig. 3. States of the agents. Magenta dots indicate switching instants.

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