Decentralized Output Synchronization of Heterogeneous Linear Systems with Fixed and Switching Topology via Self-Triggered Communication

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Abstract— **In this paper, we investigate a decentralized output synchronization problem of heterogeneous linear systems. Motivated by recent results in the literature, we develop a self-triggered output broadcasting policy for the interconnected systems. In other words, each system broadcasts its outputs only when necessary in order to achieve output synchronization. Consequently, the control signal of each system is updated based on currently available (but outdated) information received from the neighbors. These broadcasting time instants adapt to the current communication topology. For a fixed topology, our broadcasting policy yields global exponential output synchronization, and $\mathcal{L}_p$-stable output synchronization in the presence of disturbances. Employing a converse Lyapunov theorem for impulsive systems, we provide an average dwell-time condition that yields disturbance-to-state stable output synchronization in case of switching topology. The proposed approach is applicable to directed and unbalanced communication topologies. Finally, our results are corroborated by numerical simulations.**

I. INTRODUCTION

Recent years have witnessed an increasing interest in decentralized control of multi-agent systems [1]–[5]. The problem of synchronizing agents' outputs is a typical problem solved in a decentralized fashion [2], [4]. The goal of output synchronization is to achieve a desired collective behavior of multi-agent systems. Examples are formation control, flocking, consensus control, etc.

Information exchange among neighbors is instrumental for coordination [1]–[5]. According to [6], two models of *wireless networks* supporting decentralized control are typically considered in the literature. The first model is the *radio network model* that buys into the worst-case thinking – concurrent transmissions cancel each other because of interference, and potential message collision cannot be detected at a receiver's end. The second model is the *local model* that abstracts away media access issues allowing the agents to concurrently communicate with all neighbors. Clearly, the local model is too optimistic. In order to reconcile these two models, we partition the set of agents into subsets as follows: *when all agents in a subset broadcast simultaneously, the wireless network is collision free*. Basically, we do not allow agents, that belong to different partitions, to broadcast at the same time due to possibility of message collisions. In comparison with *asynchronous* wireless networks, benefits of *synchronous* wireless networks include a constant bit rate with increased Committed Information Rate (CIR) and Quality of Service (QoS), increased tolerance to interference, and low and predictable latency.

Synchronous wireless is an example of the *time-triggered* communication paradigm [7]. This paradigm excels with respect to predictability, composability, error detection and error containment. Another widely used paradigm is the *event-triggered* communication paradigm which excels with respect to flexibility and resource efficiency. As [7] suggests, a combination of the time- and event-triggered paradigm is often beneficial. In this paper, we take advantage of predictability in synchronous wireless networks to detect possible changes in the communication topology. When a receiver does not receive a message in an allotted time interval, we say that an *event* has occurred and a decentralized topology discovery algorithm is triggered [8], [9].

In order to determine when agents in different partitions should broadcast, we utilize *self-triggered* feedback developed in [10]. Essentially, based on the current topology (captured in the Laplacian matrix) and dynamics of the agents, each agent computes when to broadcast its outputs such that output synchronization is achieved. In other words, the communication between agents is neither continuous nor periodic as in [1], [2] and [5], but adapts to changes in the topology. The motivation behind self-triggering is to reduce communication and computational load without compromising stability. In addition, valid self-triggered broadcasting policies must guarantee that broadcasting instants do not accumulate in finite time which is known as Zeno behavior [11], [12]. Consequently, self-triggering eliminates the problem of arbitrary fast switching (e.g., [11], [13]) since changes in the communication topology between broadcasting instants do not impact stability.

The contributions of this paper are fourfold: a) the design of broadcasting instants for each partition of agents yielding stability for a fixed topology; b) consideration of directed and unbalanced topologies; c) the formulation of an *average dwell-time* condition leading to stability with switching topology; and d) stability analyses that take into account disturbances. We point out that [4] considers balanced and fixed topologies while [5] considers undirected and fixed topologies.

The remainder of the paper is organized as follows. Section II presents the notation, concepts from graph theory and stability notions utilized in this paper. In addition, the notion of *average dwell-time* for switched systems is presented. Section III formulates the problem of decentralized

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output synchronization with intermittent communication and switching topology. The methodology brought together to solve the problem is presented in Section IV. The case of switching topology is investigated in Section V. The proposed methodology is verified using numerical simulations in Section VI. Due to space limitations, our results are provided without proofs and the conclusion section is omitted.

II. Mathematical Preliminaries

A. Notation

To shorten the notation, we use $(x, y) := [x^\top \quad y^\top]^\top$. The dimension of a vector x is denoted n_x. We use $\|f[a,b]\|_p := \left(\int_{[a,b]} \|f(s)\|^p \mathrm{d}s\right)^{1/p}$ to denote the $\mathcal{L}_p$ norm of a Lebesgue measurable function f on $[a, b] \subset \mathbb{R}$. In this paper, $\|\cdot\|$ refers to the Euclidean norm of a vector. If the argument of $\|\cdot\|$ is a matrix, then it denotes the induced matrix 2-norm. Given $x \in \mathbb{R}^n$, we define $\overline{x} = (|x_1|, |x_2|, \ldots, |x_n|)$, where $|\cdot|$ denotes the absolute value function. When the argument of $|\cdot|$ is a set, then it denotes the cardinality of the set. Given $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, the partial order $\preceq$ is given as $x \preceq y \iff x_i \leq y_i, \forall i \in \{1, \cdots, n\}$. An n-dimensional vector with all entries 0 is denoted $\mathbf{0}_n$. The set $\mathcal{A}_n^+$ denotes the subset of all $n \times n$ matrices that are symmetric and have nonnegative entries. Finally, let $\mathbb{R}_+^n$ denote the nonnegative orthant.

B. Graph Theory

A *directed graph*, or digraph, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is a nonempty set of *nodes* (or vertices) with unique ID numbers, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of the corresponding *edges*. When the edge (i, j) belongs to $\mathcal{E}$, it means that there is an information flow from the node i to the node j. We do not allow *self-loops*, i.e., edges that connect a vertex to itself. When both (i, j) and (j, i) belong to $\mathcal{E}$, we say that the *link* between i and j is *bidirectional*. Otherwise, the link between i and j is *unidirectional*. The set of *neighbors* of the node v_i is $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ which is all nodes that the node v_i can obtain information from. A *path* in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. The *distance* between two vertices in a graph is the number of edges in a shortest path connecting them. The greatest distance between any pair of vertices is called the *diameter* of a graph and is denoted $\mathrm{diam}(\mathcal{G})$. A *cycle* in $\mathcal{G}$ is a directed path with distinct nodes except for the starting and ending node. An *inclusive cycle* for an edge is a cycle that contains the edge on its path.

Given a graph $\mathcal{G}$, the graph Laplacian matrix $L \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$ is defined as $L = [l_{ij}]$, where l_{ij} equals -1 for $j \in \mathcal{N}_i$, $|\mathcal{N}_i|$ for $j = i$, and 0 otherwise.

C. Stability Notions

Consider a hybrid (or impulsive) system

$$\Sigma \quad \left\{ \begin{array}{l} \left. \begin{array}{c} \dot{x} = f(x, \omega) \\ y = g(x, \omega) \end{array} \right\} t \in \bigcup_{i \in \mathbb{N}_0} [t_i, t_{i+1}), \\ x(t^+) = h(x(t)) \qquad t \in \mathcal{T}, \end{array} \right. \tag{1}$$

with the state $x \in \mathbb{R}^{n_x}$, output $y \in \mathbb{R}^{n_y}$ and input $\omega \in \mathbb{R}^{n_\omega}$. We assume enough regularity on f and h to guarantee existence of the solutions given by right-continuous functions $t \mapsto x(t)$ on $[t_0, \infty)$ starting from x_0 at $t = t_0$. Jumps of the state x occur at each $t \in \mathcal{T} := \{t_i : i \in \mathbb{N}\}$. The value of the state after the jump is given by $x(t^+) = \lim_{t' \searrow t} x(t')$ for each $t \in \mathcal{T}$.

Definition 1: (global exponential stability) For $\omega \equiv 0$, the equilibrium point $x = 0$ of Σ is Globally Exponentially Stable (GES) if there exist $k, l \geq 0$ such that $\|x(t)\| \leq k \exp(-l(t - t_0))\|x(t_0)\|$ for all $t \geq t_0$ and for any $x(t_0)$.

Definition 2: (input-to-state stability) The system Σ is input-to-state stable (ISS) if there exist a class-$\mathcal{KL}$ function β and a class-$\mathcal{K}_\infty$ function γ such that, for any $x(t_0)$ and every input ω, the corresponding solution $x(t)$ satisfies $\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\|\omega[t_0, t]\|_\infty)$.

Definition 3: (uniform bounded-input bounded-output stability) The system Σ is uniformly bounded-input bounded-output stable if there exists a finite constant η such that, for any t_0 and any input signal $\omega(t)$, the corresponding zero-state response (i.e., $x_0 = \mathbf{0}_{n_x}$) satisfies $\|x[t_0, t]\|_\infty \leq \eta\|\omega[t_0, t]\|_\infty$.

Definition 4: ($\mathcal{L}_p$-stability) Let $p \in [1, \infty]$. The system Σ is $\mathcal{L}_p$-stable from ω to y with gain $\gamma \geq 0$ if there exists $K \geq 0$ such that $\|y[t_0, t]\|_p \leq K\|x_0\| + \gamma\|\omega[t_0, t]\|_p$ for all $t \geq t_0$.

Definition 5: (detectability) Let $p, q \in [1, \infty]$. The state x of Σ is $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from (y, ω) to x with gain $\gamma \geq 0$ if there exists $K \geq 0$ such that $\|x[t_0, t]\|_q \leq K\|x_0\| + \gamma\|y[t_0, t]\|_p + \gamma\|\omega[t_0, t]\|_p$ for all $t \geq t_0$.

Definitions 1 and 2 are taken from [14], Definition 3 is taken from [15], while Definitions 4 and 5 are found in [16].

D. Switched Systems and Average Dwell-Time

Consider a family of systems (1) indexed by the parameter ρ taking values in a set $\mathcal{P} = \{1, 2, \ldots, m\}$. Let us define a right-continuous and piecewise constant function $\sigma : [t_0, \infty) \to \mathcal{P}$ called a *switching signal* [11]. The resulting switched system is given by

$$\Sigma_\sigma \quad \left\{ \begin{array}{l} \left. \begin{array}{c} \dot{x} = f_\sigma(x, \omega) \\ y = g(x, \omega) \end{array} \right\} t \in \bigcup_{i \in \mathbb{N}_0} [t_i, t_{i+1}), \\ x(t^+) = h_\sigma(x(t)) \qquad t \in \mathcal{T}. \end{array} \right. \tag{2}$$

For each switching signal σ and each $t \geq t_0$, let $N_\sigma(t, t_0)$ denote the number of discontinuities, called *switching times*, of σ on the open interval (t_0, t). We say that σ has *average dwell-time* τ_a if there exist two positive numbers N_0 and τ_a such that

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_a} \tag{3}$$

for every $t \geq t_0$. For a comprehensive discussion refer to [11] and [13]. In this paper, different values of σ correspond to different topologies L.

III. Problem Statement

Consider N linear systems, i.e., agents, given by

$$\dot{x}_i = A_i x_i + B_i u_i, \qquad y_i = C_i x_i, \tag{4}$$

where $x_i \in \mathbb{R}^{n_{x_i}}$ is the state, $u_i \in \mathrm{R}^{n_{u_i}}$ is the input, $y_i \in \mathbb{R}^{n_y}$ is the output of the i^{th} system, $i \in \{1, 2, \ldots, N\}$, and A_i, B_i, C_i are matrices of appropriate dimensions. The set of all agents is denoted $\mathcal{V}$; hence, $|\mathcal{V}| = N$. Motivated by [2, Chapter 2], we consider the following decentralized control policy

$$u_i = -K_i \sum_{j \in \mathcal{N}_i} [(y_i - y_j) - (d_i - d_j)] + \omega_i, \tag{5}$$

where K_i is a $n_{u_i} \times n_y$ matrix, $\mathcal{N}_i$ denotes the set of neighbors of the i^{th} system, $d_i \in \mathbb{R}^{n_y}$ is the bias term, and $\omega_i \in \mathbb{R}^{n_y}$ is the disturbance term. Next, we define the following stack vectors $x := (x_1, x_2, \ldots, x_N)$, $y := (y_1, y_2, \ldots, y_N)$, $d := (d_1, d_2, \ldots, d_N)$ and $\omega := (\omega_1, \omega_2, \ldots, \omega_N)$. Knowing the Laplacian matrix L of a communication graph $\mathcal{G}$, the closed-loop dynamic equation of (4) given the control law (5) becomes

$$\dot{x} = A^{\text{cl}} x - B^{\text{cl}} d + B^{\text{d}} \omega, \qquad y = C^{\text{cl}} x, \tag{6}$$

where

$$A^{\text{cl}} = [A^{\text{cl}}_{ij}], \quad A^{\text{cl}}_{ij} = \begin{cases} A_i - l_{ii} B_i K_i C_i, & i = j \\ -l_{ij} B_i K_i C_j, & \text{otherwise} \end{cases},$$
$$B^{\text{cl}} = [B^{\text{cl}}_{ij}], \quad B^{\text{cl}}_{ij} = -l_{ij} B_i K_i,$$
$$C^{\text{cl}} = \operatorname{diag}(C_1, C_2, \ldots, C_N), B^{\text{d}} = \operatorname{diag}(B_1, B_2, \ldots, B_N).$$

In the above expressions, A^{cl}_{ij} and B^{cl}_{ij} are matrix blocks while $\operatorname{diag}(\cdot, \cdot, \ldots, \cdot)$ denotes the block-diagonal matrix.

Assumption 1: A^{cl} is Hurwitz.

When $\omega \equiv \mathbf{0}_{n_y}$, the equilibrium of (6) is given by

$$x_{eq} = \left(A^{\text{cl}}\right)^{-1} B^{\text{cl}} d \tag{7}$$

with the corresponding output (i.e., synchronization point)

$$y_{eq} = C^{\text{cl}} x_{eq}. \tag{8}$$

Definition 6: Suppose we have a system of N agents given by (4). We say that the agents output synchronize if $\|y - y_{eq}\| \to 0$ as $t \to \infty$.

Remark 1: The above definition of output synchronization differs from the definition found in, for instance, [4] where it is required that $\|y_i - y_j\| \to 0$ as $t \to \infty$ for all $i, j \in \{1, 2, \ldots, N\}$. Our definition aims at controlling the asymptotic values y_{eq} of the outputs y regardless of initial conditions. Problems in which the asymptotic values of y depend on initial conditions are characterized by matrices A^{cl} that have an eigenvalue at the origin. These problems are the subject of our forthcoming publications. In our definition, notice that one can change y_{eq} by changing d. For example, one can change formations by changing d.

It is well known that substitutions $x' = x - x_{eq}$ and $y' = y - y_{eq}$ transform (6) into the equivalent system

$$\dot{x}' = A^{\text{cl}} x' + B^{\text{d}} \omega, \qquad y' = C^{\text{cl}} x', \tag{9}$$

such that $x'_{eq} = 0$ is the equilibrium point when $\omega \equiv \mathbf{0}_{n_y}$. From Assumption 1 and [14, Corollary 5.2.], we infer that the closed-loop system (9) is $\mathcal{L}_p$-stable from ω to y' for each $p \in [1, \infty]$.

Since we do not consider continuous feedback in (5), the control signal becomes

$$u_i = -K_i \sum_{j \in \mathcal{N}_i} [(\hat{y}_i - \hat{y}_j) - (d_i - d_j)] + \omega_i, \tag{10}$$

where $\hat{y}_j$ is the most recently transmitted value of the output of the j^{th} agent. Let $\mathcal{T}_i := \{t_i^j : j \in \mathbb{N}\}$ denote the set of broadcasting time instants of the i^{th} agent and $\mathcal{T} := \cup_{i=1}^N \mathcal{T}_i$. In order to account for the fact that outdated $\hat{y}_i$'s are used in control law (5) and not the actual outputs y_i's, we introduce the output error vector

$$e := \hat{y} - y. \tag{11}$$

The above expression uses $\hat{y} := (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_N)$. Taking e into account, the closed-loop dynamics (9) become

$$\dot{x}' = A^{\text{cl}} x' + B^{\text{cl}} e + B^{\text{d}} \omega. \tag{12}$$

Since $\dot{\hat{y}} = 0$ and $\dot{y}_{eq} = 0$, the corresponding output error dynamics are

$$\dot{e} = -C^{\text{cl}} \dot{x}'. \tag{13}$$

Problem 1: Partition the set of agents $\mathcal{V}$ into subsets $\mathcal{P}_i$'s with the following property: when all agents in each $\mathcal{P}_i$ broadcast simultaneously, message collisions are avoided.

Problem 2: Given a fixed topology, design sets of broadcasting instants $\mathcal{T}_i$, $i \in \{1, 2, \ldots, N\}$, such that the outputs of agents synchronize in the sense of Definition 6.

Problem 3: Find conditions that preserve output synchronization under switching communication topology.

IV. Methodology

A. Decentralized Topology Discovery for Directed Graphs

In order to solve Problem 1, each agent has to know the communication topology, i.e., the graph Laplacian matrix L. This problem is known as topology discovery and has been an active area of research (e.g., [8] and [9]). In this paper we implement the approach from [8] due to its applicability to directed graphs. This approach converges in finite time Δ after the network topology stops changing. Δ is proportional to $\operatorname{diam}(\mathcal{G})$. Since we consider a finite number of agents N, there exists an upper bound on Δ, denoted Δ_u, for all admissible topologies. The admissible topologies are those that satisfy Assumptions 1 and 2.

Assumption 2: All unidirectional links have an inclusive cycle.

Assumption 2 is the main assumption in [8]. In order to simplify the exposition of this paper, we take $\Delta_u = 0$.

Remark 2: Notice that leader-follower topologies do not satisfy Assumption 2. In case of fixed communication topologies, Assumption 2 can be omitted as this assumption is needed only for decentralized topology discovery. Thus, the work presented herein is applicable to time-invariant leader-follower topologies as well.

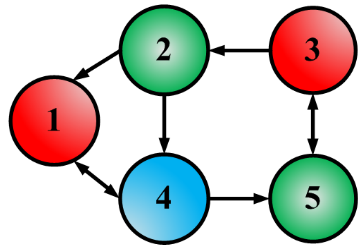

Fig. 1. The graph partition $\mathcal{P}_1 = \{1, 3\}$, $\mathcal{P}_2 = \{2, 5\}$ and $\mathcal{P}_3 = \{4\}$.

B. Partitioning the Agents

After obtaining L using the approach from [8], we have to partition the set $\mathcal{V}$ in order to avoid message collisions according to the *radio network model*. In other words, we want to allow simultaneous broadcast of agents that have no common receivers and are not receivers themselves at that particular time instants. Notice that, if one is not concerned with message collisions, then there is no need to partition the agents.

Consider the graph depicted in Figure 1. The corresponding graph Laplacian matrix is

$$L_1 = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}. \tag{14}$$

This graph satisfies Assumption 2. Next, we partition $\mathcal{V} = \{1, 2, 3, 4, 5\}$ in Figure 1 using Algorithm 1. In Algorithm 1, the element-wise product of the i^{th} and the j^{th} column of L is denoted $L(i).L(j)$. The input to the algorithm is L, and the outputs are subsets $\mathcal{P}_i$. The number of nonempty $\mathcal{P}_i$'s is $T \leq N$, and we prune empty $\mathcal{P}_i$'s.

Algorithm 1 Graph partitioning algorithm.

```
P_i ← {∅} for all i ∈ {1,...,N}; k ← 0
for i = 1 to N do
    if i ∉ P_m for every m ∈ {1,...,N} then
        k ← k + 1
        P_k ← P_k ∪ {i}
        for j = i + 1 to N do
            if L(i).L(j) = 0_N for all i ∈ P_k then
                P_k ← P_k ∪ {j}
            end if
        end for
    end if
end for
```

C. Designing Broadcasting Instants

In order to solve Problem 2, we use the extensions of [16] reported in [10]. The details of [10] are out of scope of this paper. In what follows, we provide only the scheduling protocol considered herein and adapt results from [10] to this specific protocol. In other words, even though the framework presented in this subsection is applicable to the larger group of *uniformly persistently exciting scheduling protocols* [16], we do not pursue that direction in this paper.

Protocol 1: The agents from $\mathcal{P}_{[(i+1) \mod T]+1}$ broadcast their outputs τ seconds after the agents from $\mathcal{P}_{[i \mod T]+1}$ have broadcast their outputs, where $\mod$ is the module operator.

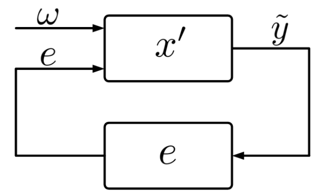

Fig. 2. Interconnection of the nominal and the output error dynamics.

Notice that elements of the set $\mathcal{T}_{[(i+1) \mod T]+1}$ are equal to elements $\mathcal{T}_{[i \mod T]+1}$ increased by τ. The impact of broadcasting agents' outputs is as follows:

Property 1: If the i^{th} agent broadcasts at time t, the corresponding components of e reset to zero while other components remain unchanged, i.e.,

$$\left.\begin{aligned} e^+_{(i-1)n_y+1}(t) = \ldots = e^+_{in_y}(t) = 0, \\ e^+_j(t) = e_j(t), \end{aligned}\right\} \tag{15}$$

for all $j \in \{1, \ldots, Nn_y\} \setminus \{(i-1)n_y + 1, \ldots, in_y\}$, where the set difference is denoted $\setminus$.

Now, let us interconnect dynamics (12) and (13) and employ the small-gain theorem [14]. To this end, we upper bound the output error dynamics (13) as follows:

$$\dot{\bar{e}} = \overline{-C^{\text{cl}}(A^{\text{cl}}x' + B^{\text{cl}}e + B^{\text{d}}\omega)} \preceq A^*\bar{e} + \tilde{y}(x', \omega), \tag{16}$$

where

$$A^* = [a^*_{ij}] := \max\{|c^*_{ij}|, |c^*_{ji}|\}, \tag{17}$$

$$\tilde{y}(x, \omega) := \overline{C^{\text{cl}}(A^{\text{cl}}x + B^{\text{d}}\omega)}. \tag{18}$$

In (17), we use $-C^{\text{cl}}B^{\text{cl}} = [c^*_{ij}]$. Notice that $A^* \in \mathcal{A}^+_{n_e}$ and $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\omega} \to \mathbb{R}^{n_e}_+$ is a continuous function. With this choice of A^* and $\tilde{y}$, the upper bound (16) holds for all $(x', e, \omega) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_\omega}$ and all $t \in \mathbb{R}$.

Theorem 1: Suppose that Protocol 1 is implemented. In addition, suppose that $\tau \in (0, \tau^*)$, where $\tau^* := \frac{\ln(2)}{\|A^*\|T}$. Then, the output error system (13) is $\mathcal{L}_p$-stable from $\tilde{y}$ to e for any $p \in [1, \infty]$ with gain

$$\gamma_e = \frac{T\exp(\|A^*\|(T-1)\tau)(\exp(\|A^*\|\tau) - 1)}{\|A^*\|(2 - \exp(\|A^*\|T\tau))}. \tag{19}$$

Next, take (e, ω) to be the input and $\tilde{y}$, obtained in Theorem 1, to be the output of the dynamics (12). For given $p \in [1, \infty]$, the corresponding $\mathcal{L}_p$-gain from (e, ω) to $\tilde{y}$ is denoted γ. Hence, systems (12) and (13) are interconnected according to Figure 2.

Theorem 2: Suppose that Protocol 1 is implemented. If the interbroadcasting interval τ in (19) is such that $\gamma\gamma_e < 1$, then the interconnection in Figure 2 is $\mathcal{L}_p$-stable from ω to $(e, \tilde{y})$ for given $p \in [1, \infty]$.

Remark 3: Notice that $\gamma_e(\tau)$ in (19) is a monotonically increasing function of $\tau \in [0, \tau^*)$. In addition, notice that $\gamma_e(0) = 0$. By the assumption of Theorem 2, we know that $\gamma < \infty$. Since our goal it to design τ such that $\gamma\gamma_e(\tau) < 1$, we first find τ' such that $\gamma\gamma_e(\tau') = 1$, and then compute $\tau = \kappa\tau'$, where $\kappa \in (0, 1)$. Due to monotonicity of $\gamma_e(\tau)$, the obtained τ' is strictly positive; hence, $\tau = \kappa\tau'$ is strictly positive. Consequently, the unwanted Zeno behavior [12] is avoided. In other words, our approach does not yield continuous feedback that is impossible to implement with digital technology.

Remark 4: Because this paper investigates linear time-invariant agents (4) and control laws (5), the upper bound (16) holds globally. Consequently, γ_e in (19) does not depend on (x', e, ω). Hence, in case the topology is fixed, the rule $\tau = \kappa\tau'$ yields periodic broadcasting for the given topology. This is the time-triggered feature of our approach. Next, any change of the topology generates an event that triggers a recomputation of τ. This is the event-triggered feature of our approach. However, scenarios with a time-varying κ or the upper bound (16) that holds only locally would yield a time-varying τ even for a fixed topology (refer to [10] for more details). Having that said, our methodology is surely an instant of self-triggering. Lastly, notice that τ is independent of y_{eq}, i.e., the agents do not need to know the synchronization point y_{eq} ahead of time which would in turn compromise the decentralized nature of our approach.

From Assumption 1 and [14, Corollary 5.2.], we infer that for (12) there exist $K_d, \gamma_d \geq 0$ such that $\|x'[t_0, t]\|_p \leq K_d\|x'_0\| + \gamma_d\|(e, \omega)[t_0, t]\|_p$ for any $p \in [1, \infty]$. Consequently, x' is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $(e, \omega, \tilde{y})$ for any $p \in [1, \infty]$.

Corollary 1: Assume that the conditions of Theorem 2 are met. Then, output synchronization of systems given by (4) is $\mathcal{L}_p$-stable from ω to (e, x') for given $p \in [1, \infty]$.

Corollary 2: Assume that the conditions of Theorem 2 are met. Then, output synchronization of systems given by (4) is ISS from ω to (e, x').

V. Stability under Switching Topology

The switched system with impulsive effects we are considering herein is

$$\begin{bmatrix} \dot{x}' \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_\sigma^{cl} & B_\sigma^{cl} \\ -C^{cl}A_\sigma^{cl} & -C^{cl}B_\sigma^{cl} \end{bmatrix} \begin{bmatrix} x' \\ e \end{bmatrix} + \underbrace{\begin{bmatrix} B^d \\ -C^{cl}B^d \end{bmatrix}}_{B_\omega} \omega, t \notin \mathcal{T},$$

$$\left.\begin{aligned} x'^{+}(t) &= x'(t) \\ e^{+}(t) &= \Gamma_i e(t) \end{aligned}\right\} \quad t \in \mathcal{T}, \tag{20}$$

where matrix Γ_i implements Property 1 at the i^{th} broadcasting instant. To shorten the notation, we use $z := (x', e)$.

A. Switching without Disturbances

After setting $\omega \equiv \mathbf{0}_{n_y}$ in (20), we obtain the following result:

Theorem 3: Suppose that the conditions of Corollary 1 hold and $\omega \equiv \mathbf{0}_{n_y}$. In addition, assume that L is fixed. Then, the equilibrium point $(e, x') = 0$ of the closed-loop system (12) and (13) is GES.

Next, notice that the equilibrium point x_{eq} given by (7) is a function of L. In other words, different communication topologies result in different x_{eq}, i.e., different x'. In order to apply results from [13] and [11], x_{eq} must be the same for all admissible topologies. This can be achieved by adapting d in (7) such that x_{eq} is constant as the topology changes or one can simply use $d = \mathbf{0}_{n_d}$ yielding $x_{eq} = \mathbf{0}_{n_x}$. For the sake of simplicity, we use $d = \mathbf{0}_{n_y}$ herein. Consequently, $x' = x$ holds so we use x instead of x' in the remainder of the paper.

According to Theorem 3, each subsystem in $\mathcal{P}$ is GES. Let us now apply [17, Theorem 15.3.] to each subsystem in $\mathcal{P}$. From (20) we infer that the flow and jump maps are Lipschitz continuous and are zero at zero. In addition, jump times t_i's are predefined (i.e., time-triggered and do not depend on the actual solution of the system), and such that $0 < t_1 < t_2 < \ldots < t_i$ and $\lim_{i\to\infty} t_i = \infty$ hold. Consequently, all conditions of [17, Theorem 15.3.] are met. Hence, there exist functions $V_\rho : \mathbb{R} \times \mathbb{R}^{n_x+n_e} \to \mathbb{R}$, $\rho \in \mathcal{P}$, that are right-continuous in t and Lipschitz continuous in z, and satisfy the following inequalities

$$c_{1,\rho}\|z\|^2 \leq V_\rho(t, z) \leq c_{2,\rho}\|z\|^2, \qquad t \geq t_0, \tag{21}$$

$$D_\rho^+ V_\rho(t, z) \leq -c_{3,\rho}\|z\|^2, \qquad t \notin \mathcal{T}, \tag{22}$$

$$V_\rho(t^+, z^+) \leq V_\rho(t, z), \qquad t \in \mathcal{T}, \tag{23}$$

for all $z \in \mathbb{R}^{n_x+n_e}$, where $c_{1,\rho}$, $c_{2,\rho}$ and $c_{3,\rho}$ are positive constants. These constants are readily obtained once k and l from Definition 1 are known (see the proof of [17, Theorem 15.3.]). In the above inequalities, $D_\rho^+ V_\rho(t, z)$ denotes the upper right derivative of function V_ρ with respect to the solutions of the ρ^{th} system. The upper right derivative of V_ρ is given by $D_\rho^+ V_\rho(t, z) := \limsup_{h\to 0, h>0} \left(\frac{1}{h}[V_\rho(t + h, z(t+h)) - V_\rho(t, z(t))]\right)$, where $z(t)$, $t \geq t_0$, denotes the trajectory of the ρ^{th} system. We now rewrite (21) and (22) as follows

$$c_1\|z\|^2 \leq V_\rho(t, z) \leq c_2\|z\|^2, \qquad t \geq t_0, \tag{24}$$

$$D_\rho^+ V_\rho(t, z) \leq -2\lambda_0 V_\rho(t, z), \qquad t \notin \mathcal{T}, \tag{25}$$

$$V_\rho(t, z) \leq \mu V_\varrho(t, z), \qquad t \geq t_0, \tag{26}$$

for all $z \in \mathbb{R}^{n_x+n_e}$ and all $\rho, \varrho \in \mathcal{P}$, where

$$c_1 = \min_{\rho\in\mathcal{P}} c_{1,\rho} > 0, \qquad c_2 = \max_{\rho\in\mathcal{P}} c_{2,\rho} > 0,$$

$$\lambda_0 = \min_{\rho\in\mathcal{P}} \frac{c_{3,\rho}}{2c_{1,\rho}} > 0, \qquad \mu = \max_{\rho,\varrho\in\mathcal{P}} \frac{c_{2,\rho}}{c_{1,\varrho}} > 0.$$

Notice that $\mu > 1$ in the view of interchangeability of ρ and ϱ in (26). Following ideas from [11] and [13], we obtain the following result:

Theorem 4: Consider the family of m systems for which (23), (24), (25) and (26) hold. Then the corresponding switched system (20) is GES uniformly in t_0 for every switching signal σ with average dwell-time

$$\tau_a > \frac{\ln \mu}{2\lambda_0} \tag{27}$$

and N_0 arbitrary.

Remark 5: Recall that changes of the topology in $[t_i, t_{i+1})$, where $t_i, t_{i+1} \in \mathcal{T}$, remain unnoticed until t_{i+1} (or even later). Therefore, if $\min_{i\in\mathcal{P}} \tau_i \geq \tau_a$, then we effectively have that the switched system (20) is GES uniformly in t_0 for any switching signal. Obviously, we want to obtain τ_i's in Subsection IV-C as large as possible. This is yet another motivation for developing self-triggered control policies.

B. Switching with Disturbances

Notice that (20) can be interpreted as a linear time-varying impulsive system. From Theorem 4 we infer that the state transition matrix $\Phi(t, t_0)$ of (20) satisfies

$$\|\Phi(t, t_0)\| \leq k \exp(-l(t - t_0)), \tag{28}$$

where $k = \sqrt{\frac{c_2}{c_1}\mu^{N_0}}$ and $l = \lambda$ for some $\lambda \in (0, \lambda_0)$. For the explicit form of state transition matrices of linear time-varying impulsive systems refer to [17, Chapter 3]. From the corresponding variation of constants formula (see [17, Chapter 3]) $z(t) = \Phi(t, t_0)z(0) + \int_{t_0}^{t} \Phi(t, s)B_\omega \omega(s)\mathrm{d}s$, and (28), we obtain $\|z(t)\| \leq k \exp(-l(t - t_0))z(0) + bk \int_{t_0}^{t} \exp(-l(t - s))\|\omega(s)\|\mathrm{d}s$, where $\|B_\omega\| \leq b$. Since $t \geq t_0$, therefore $\int_{t_0}^{t} \exp(-l(t - s))\mathrm{d}s \leq 1/l$ for any t_0. Using [15, Theorem 12.2], we infer that (20) is *uniformly bounded-input bounded-state* stable which in turn implies ISS (refer to [2, Theorem 2.35 & Remark 2.36] for more).

Theorem 5: Assume that Theorem 4 holds so that the system (20) is GES uniformly in t_0 for every switching signal σ with average dwell-time (27). Then, output synchronization of systems given by (4) is ISS from ω to (e, x).

VI. Example

Consider the following five agents:

$$A_1 = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}, C_1 = \begin{bmatrix} -4 & 1 \end{bmatrix}, \quad K_1 = 2,$$

$$A_2 = \begin{bmatrix} 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \end{bmatrix}, C_2 = \begin{bmatrix} -5 \end{bmatrix}, \quad K_2 = 4,$$

$$A_3 = \begin{bmatrix} -2 & -5 \\ 1 & -1 \end{bmatrix}, B_3 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, C_3 = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad K_3 = -2,$$

$$A_4 = \begin{bmatrix} -3 & -1 \\ 5 & -3 \end{bmatrix}, B_4 = \begin{bmatrix} -4 \\ -2 \end{bmatrix}, C_4 = \begin{bmatrix} 0 & -4 \end{bmatrix}, \quad K_4 = 4,$$

$$A_5 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B_5 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, C_5 = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}, \quad K_5 = 5.$$

In addition to topology L_1 given by (14), we consider another topology given by

$$L_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}. \tag{29}$$

Applying Algorithm 1 to L_2, we obtain $\mathcal{P}_1 = \{1, 3\}$, $\mathcal{P}_2 = \{2, 4\}$ and $\mathcal{P}_3 = \{5\}$.

Let us consider the case $p = 2$ and apply [14, Theorem 5.4]. Using the MATLAB function $\mathrm{norm}(\cdot, \mathrm{inf})$, the corresponding $\mathcal{L}_2$-gain from (e, ω) to $\tilde{y}$ is readily obtained: $\gamma_1 = 132$. Solving (19) with $T = 3$ such that $\gamma_1 \gamma_{e,1} < 1$ yields $\tau_1 = 1.1 * 10^{-3}$ s for $\kappa = 0.999$. This corresponds to broadcasting frequency of 307 Hz for each agent. The same steps for L_2 yield $\gamma_2 = 267$, $\tau_2 = 8.7 * 10^{-4}$ s and broadcasting frequency 383 Hz for each agent.

In order to verify results of Section V, we toggle between topologies L_1 and L_2. Numerical results are provided in Figure 3. We choose $\omega_i(t) = 5t_{[0,4.4)} - 5t_{[4.4,8.8)} + 0t_{[8.8,13]}$, $i \in \{1, \ldots, n_d\}$, where $t_{\mathcal{I}}$ is the indicator function on an interval $\mathcal{I}$. In other words, $t_{\mathcal{I}} = t$ when $t \in \mathcal{I}$ and zero otherwise.

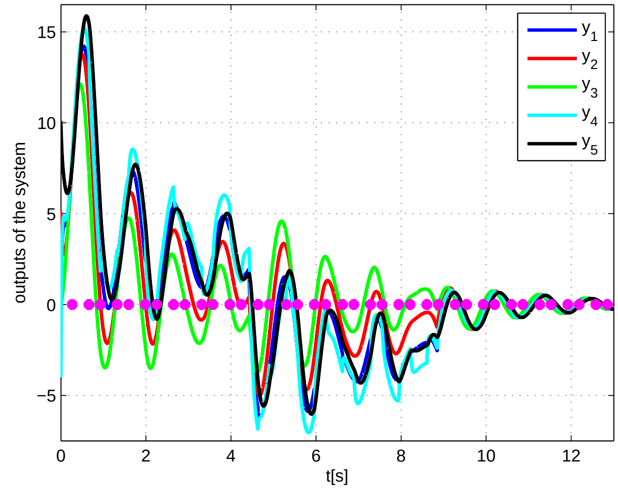

Fig. 3. Outputs of the agents for a scenario with disturbance. Magenta dots indicate switching instants.

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